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Matrix :

A matrix is an ordered rectangular array of numbers or functions.

The numbers or functions are called the elements or the entries of the matrix. It is denoted by capital letter. It is denoted by ( ), [ ], ‖ ‖.

$$A = \begin{bmatrix} 8 & 10 \\ 10 & 15 \\ 18 & 20 \end{bmatrix}$$

Order of a matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as an m by n matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} & \cdots a_{2n} \\ \vdots & \vdots & & & \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} & \cdots a_{in} \\ \vdots & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} & \cdots a_{mn} \end{bmatrix}$$

Q. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

⇒

In general a 3×2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now $a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore. $a_{11} = \frac{1}{2}|1 - 3\times 1| = 1.$

$a_{12} = \frac{1}{2}|1 - 3\times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3\times 1| = \frac{1}{2}$

$a_{22} = \frac{1}{2}|2 - 3\times 2| = \frac{4}{2} = 2.$

$a_{31} = \frac{1}{2}|3 - 3\times 1| = 0.$

$a_{32} = \frac{1}{2}|3 - 3\times 2| = \frac{3}{2}.$

Hence the required matrix is given by

$$A = \begin{bmatrix} 1 & 5/2 \\ 1/2 & 2 \\ 0 & 3/2 \end{bmatrix}$$

Types of Matrices:

(i) Column matrix

A matrix is said to be a column matrix if it has only one column.

$$A = \begin{array}{c} \begin{bmatrix} 8 \\ 15 \\ 20 \end{bmatrix}_{3\times 1} \\ \downarrow \\ C_1 \end{array} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \\ \\ \\ \end{array}$$

(ii) Row matrix :

A matrix is said to be a row matrix if it has only one row.

$$A = \begin{bmatrix} 1 & 4 & 6 \end{bmatrix}_{1\times 3} \rightarrow R_1$$
$$\quad \downarrow \quad \downarrow \quad \downarrow$$
$$\quad C_1 \quad C_2 \quad C_3$$

(iii) Square matrix:

A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix.

$$A = \begin{bmatrix} 8 & 13 & 5 \\ 3 & 2 & 15 \\ 18 & 20 & 13 \end{bmatrix}_{3\times 3}$$

$\boxed{\text{where } m = n}$

(iv) Diagonal matrix.

A square matrix $B = [b_{ij}]_{m\times n}$ is said to be a diagonal matrix if all its non-diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

(v) Scalar matrix

A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal.

$$A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(vi) Identity matrix:

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vii) Zero matrix:

A matrix is said to be zero matrix or null matrix if all its elements are zero.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equality of matrices.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if.

(i) they are of the same order.

(ii) each element of A is equal to the corresponding element of B, ~~tha~~ that is $a_{ij} = b_{ij}$ for all i and j.

$\therefore \boxed{A = B}$ $\quad A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

Operations on Matrices.

Addition of matrices.

Q. $A = \begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix}$, $B = \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

$$A + B = \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ Ans

Properties of matrix addition.

(i) Commutative Law:

If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then $A + B = B + A$.

eg. → $[a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}]$.

(ii) Associative law:

For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, then

$$(A + B) + C = A + (B + C)$$

$$([a_{ij}] + [b_{ij}]) + [c_{ij}] = [a_{ij}] + ([b_{ij}] + [a_{ij}])$$

(iii) Existence of additive identity.

~~Let A = [aij] be an mxn matrix, then we have another matrix as -A = [-aij] mxn such that A+~~

Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, O is the additive identity for matrix addition.

(iv) The Existence of additive inverse:

Let $A = [a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = (-A) + A = O$. So $-A$ is the additive inverse of A or negative of A.

$$8 + (-8) = 0 = (-8) + 8.$$

Multiplication of Matrices:

$$A \times B = \begin{bmatrix} 3 & 8 \\ 2 & 10 \end{bmatrix} \times \begin{bmatrix} 5 & 7 \\ 4 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 5 + 8 \times 4 & 3 \times 7 + 8 \times 15 \\ 2 \times 5 + 10 \times 4 & 2 \times 7 + 10 \times 15 \end{bmatrix}$$

$$= \begin{bmatrix} 15+32 & 21+120 \\ 10+40 & 14+150 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 141 \\ 50 & 164 \end{bmatrix}$$

Definition.

The product of two matrices, A and B defined if the number of columns of A is equal to the no. of rows of B. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$.

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Q. $A = \begin{bmatrix} 3 & 2 \\ 8 & 6 \end{bmatrix}_{2\times 4}$

$B = \begin{bmatrix} 2 & 8 & 3 & 4 \\ 3 & 4 & 8 & 7 \end{bmatrix}_{2\times 4}$.

$A \times B = \begin{bmatrix} 6+6 & 24+8 & 9+16 & 12+14 \\ 16+18 & 64+24 & 24+48 & 32+42 \end{bmatrix}$

$A \times B = \begin{bmatrix} 12 & 32 & 25 & 26 \\ 34 & 88 & 72 & 74 \end{bmatrix}_{2\times 4}$.

Transpose of matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix}$$

$$A' \text{ or } A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 3 & 28 \\ 3 & 18 & 13 \end{bmatrix}$$

$$A' = \begin{bmatrix} 8 & 3 \\ 3 & 18 \\ 28 & 13 \end{bmatrix}$$

Definition.

If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or (A^T).

In other words, if $A = [a_{ij}]_{m \times n}$, then

$$A' = [a_{ji}]_{n \times m}.$$

$$A = \begin{bmatrix} 8 & 3 & 18 \\ 3 & 28 & 13 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 8 & 3 \\ 3 & 28 \\ 18 & 13 \end{bmatrix}$$

Property of Transpose of matrices.

For any matrices A and B is of suitable orders, we have.

(i) $A = (A')'$

If $A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

Now $A' = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 6 & 8 \end{bmatrix}$

then $(A')' = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

(ii) $(KA)' = KA'$ where K is constant.

$$A = \begin{bmatrix} 5 & 6 & 13 \\ 3 & 8 & 10 \end{bmatrix}$$

$$KA = \begin{bmatrix} 5K & 6K & 13K \\ 3K & 8K & 10K \end{bmatrix}$$

$$(KA)' = \begin{bmatrix} 5K & 3K \\ 6K & 8K \\ 13K & 10K \end{bmatrix}$$

$$= k \begin{bmatrix} 5 & 3 \\ 6 & 8 \\ 13 & 10 \end{bmatrix}$$

$$= KA'$$

Then $A' = \begin{bmatrix} 5 & 3 \\ 6 & 8 \\ 13 & 10 \end{bmatrix}$

$\Rightarrow \boxed{(KA)' = KA'}$

3. $(A+B)' = A' + B'$

$A = \begin{bmatrix} 13 & 8 \\ 5 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$

$A' = \begin{bmatrix} 13 & 5 \\ 8 & 10 \end{bmatrix}$, $B' = \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix}$

$$L.H.S = \begin{bmatrix} 13 & 8 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 13 \\ 11 & 17 \end{bmatrix}$$

$$R.H.S = A' + B$$

$$= \begin{bmatrix} 13 & 5 \\ 8 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 11 \\ 13 & 17 \end{bmatrix}$$

$R.H.S = L.H.S$ proved.

4. $(AB)' = B'A'$

$$A = \begin{bmatrix} 5 & 6 \\ 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 15 \\ 2 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 5 & 3 \\ 6 & 7 \end{bmatrix}, \quad B' = \begin{bmatrix} 10 & 2 \\ 15 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 10 & 15 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 50+12 & 75+24 \\ 30+14 & 45+28 \end{bmatrix} = \begin{bmatrix} 62 & 99 \\ 44 & 73 \end{bmatrix}$$

$$L.H.S = (AB)' = \begin{bmatrix} 62 & 44 \\ 99 & 73 \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} 10 & 2 \\ 15 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 50+12 & 30+14 \\ 75+24 & 45+28 \end{bmatrix} = \begin{bmatrix} 62 & 44 \\ 99 & 73 \end{bmatrix}$$

$L.H.S = R.H.S$ proved.

Theorem 1:

For any square matrix A with real numbers entities, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Let $P = A + A'$

Now $P' = (A + A')'$

$\Rightarrow P' = A' + (A')'$

$\Rightarrow P' = A' + A$ (For commutative law)

$\Rightarrow P' = P$

$\Rightarrow P = P'$

Hence $A + A'$ is symmetric Matrix.

Let $Q = A - A'$

Now $Q' = (A - A')'$

$\Rightarrow Q' = A' - (A')'$

$\Rightarrow Q' = A' - A$

$\Rightarrow Q' = -(-A' + A)$

$\Rightarrow Q' = -(A - A')$ (From commutative law)

$\Rightarrow Q' = -Q$

Therefore $A - A'$ is a skew symmetric matrix.

Theorem 2:

Any square matrix can be expresed as the sum of a symmetric and a skew symmetric matrix.

Let A be a square matrix, then we can write:

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A').$$

Q.10 $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ $A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$ Ans

Elementary operation (Transformation) of Matrix.

1. The interchange of any two rows or two columns. Symbollically the interchange of ith and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example :-

(a) $$A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array}$$

$$R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 11 & 18 \\ 8 & 3 \end{bmatrix}$$

(b) $$A = \begin{array}{cc} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ C_1 & C_2 \end{array} \end{array}$$

$$C_1 \leftrightarrow C_2$$

$$A = \begin{bmatrix} 3 & 8 \\ 18 & 11 \end{bmatrix}$$

2. The multiplication of the elements of any row or column by a non zero number.

Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \rightarrow kR_i$.

$$\boxed{C_i \rightarrow kC_i}$$

(a)
$$A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array}$$

$$R_2 \rightarrow kR_2$$

$$A = \begin{bmatrix} 8 & 3k \\ 11 & 18k \end{bmatrix}$$

where k is non-zero constant.

(b)
$$A = \begin{array}{c} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ C_1 & C_2 \end{array} \end{array}$$

$$\boxed{C_1 \rightarrow kC_1}$$

$$A = \begin{bmatrix} 8k & 3 \\ 11k & 18 \end{bmatrix}$$

where k is non-zero constant.

3. The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by K is denoted by $R_i \to R_i + kR_j$.

Example :-

(a) $A = \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \end{array}$

$$R_1 \to R_1 + kR_2$$

$$A = \begin{bmatrix} 8+11K & 3+18K \\ 11 & 18 \end{bmatrix}$$

(b) $A = \begin{array}{c} \begin{bmatrix} 8 & 3 \\ 11 & 18 \end{bmatrix} \\ \begin{matrix} \downarrow & \downarrow \\ C_1 & C_2 \end{matrix} \end{array}$

$$C_2 \to C_2 + KC_1$$

$$A = \begin{bmatrix} 8 & 3+8K \\ 11 & 18+11K \end{bmatrix}$$

Inverse of matrices.

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that $AB = BA = I$, then B is called the inverse matrix of A and it is denoted by A^{-1}.

eg. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2\times 2 + 3\times -1 & 2\times 3 + 2\times 3 \\ 1\times 2 + 2\times -1 & 1\times 3 + 2\times 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 4-3 & -6+6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem 4.

If A and B are ~~matrices~~ invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

We know that :-

$$(AB)(AB)^{-1} = I$$

Multiplying A^{-1} on both side, we have.

$$\Rightarrow \quad A^{-1}(AB)(AB)^{-1} = A^{-1} \cdot I$$

$$\Rightarrow \quad (A^{-1}A)B(AB)^{-1} = A^{-1}$$

$$\Rightarrow \quad IB(AB)^{-1} = A^{-1}$$

$$\Rightarrow \quad B(AB)^{-1} = A^{-1}$$

Multiplication of B^{-1} on both side, we have,

$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$I(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{proved.}$$

Q.1.) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

We know that

$$A = AI$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

By Row transformation.

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow \frac{1}{5} \times R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/5 & 1/5 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} A$$

Therefore, $A^{-1} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}$ Ans.

Determinants

Definition.

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i,j)^{th}$ element of A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A

is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$.

It is denoted by $\det(A)$, $|A|$ or Δ.

Note :-

For matrix A, $|A|$ is read as determinant of A and not modulus of A.

Determinant of a matrix of order one :-

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

$$|A| \text{ or } \det(A) \text{ or } \Delta = |a| = a.$$

Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2

then the determinant of A is defined as :

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}\, a_{22} - a_{21}\, a_{12}$$

Q $A = \begin{vmatrix} 8 & 5 \\ 10 & 1 \end{vmatrix}$

Then $|A| = 8 - 50 = -42$ Ans.

Determinant of a matrix of order 3×3.

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below:-

Consider the determinant of square matrix

$$A = [a_{ij}]_{3\times 3}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1, we have:

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22} \cdot a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} \cdot a_{31})$$
$$+ a_{13}(a_{21} a_{32} - a_{22} a_{31}).$$

Q. $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along R_1, we have

$\Rightarrow (-1)^{1+1} a_{11}(a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - a_{12}(a_{21} a_{33} - a_{23} \cdot a_{31})$
$+ a_{13}(a_{21} a_{32} - a_{22} a_{31})$

$\Rightarrow 3(0.0 - (5)) - (-1)(0.0 - (-3)) + (-2)(0 - 0)$

$\Rightarrow 3 \times -5 + 1 \times 3 + (-2)$

$\Rightarrow -15 + 3 - 2$

$\Rightarrow -14$ Ans.

Properties of determinant.

i) The value of determinant remain unchanged if its row and column are interchanged.

$$A = \begin{bmatrix} 8 & 3 & 8 \\ 1 & 2 & 8 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= 8\begin{bmatrix} 2 & 8 \\ 3 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 8 \\ 5 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= 8(2-24) - 3(1-40) + 8(3-10)$$
$$= 8(-22) - 3(-39) + 8(-7)$$
$$= -176 + 117 - 56$$
$$= -115$$

$$A = \begin{bmatrix} \cancel{8} & \cancel{3} & \cancel{8} \\ \cancel{1} & \cancel{2} & \cancel{8} \\ \cancel{5} & \cancel{3} & \end{bmatrix}$$

Note: If follow from above property that A is square matrix, then $\det(A) = \det(A')$ where A' = transpose of A.

ii) If any two rows of a determinant are interchanged then sign of determinant changes.

iii) If any two rows or columns of a determinant ~~changes~~ are same then the value of determinant is zero.

$$|A| = \begin{vmatrix} 8 & 5 & 6 \\ 3 & 1 & 7 \\ 8 & 5 & 6 \end{vmatrix}$$

$$\begin{aligned} |A| &= 8(6-35) - 5(18-56) + 6(15-8) \\ &= 8(-29) - 5(-38) + 6(7) \\ &= -232 + 190 + 42 \\ &= -232 + 232 \\ &= 0 \text{ Ans.} \end{aligned}$$

iv) If each element of a row (or column) of determinant is multiplied by a constant 'K' then the value of determinant is multiplied by K.

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

$$\begin{aligned} |A| &= 3(32-35) - 3(24-30) + 4(21-24) \\ &= 3(-3) - 3(-6) + 4(-3) \\ &= -9 + 18 - 12 \\ &= -3 \end{aligned}$$

$R_1 \rightarrow KR_1$

$$|A| = \begin{vmatrix} 3k & 3k & 4k \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$$

$$|A| = 3k(32-35) - 3k(24-30) + 4k(21-24)$$
$$|A| = 3k(-3) - 3k(-6) + 4k(-3)$$
$$= -9k + 18k - 12k$$
$$= -3k \text{ Ans.}$$

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Property - 5

If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Example:-

$$A = \begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Q. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

$\Rightarrow$ $L.H.S = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

$\Rightarrow \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

$\Rightarrow \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + 2 \begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$

($R_1 = R_2$ in left), ($R_2 = R_3$ in right) (Property-3)

$\Rightarrow 0 + 2 \times 0$

$= 0 + 0 = 0.$

Property - 6.

If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remain the same.

i.e. the value of determinant remain same if we apply the operation.

$$R_i \to R_i + KR_j \quad \text{OR} \quad C_i \to C_i + KC_j$$

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

and $\Delta_1 = \begin{vmatrix} a_1 + Kc_1 & a_2 + Kc_2 & a_3 + Kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Where Δ_1 is obtained by the operation

$$R_1 \to R_1 + KR_3$$

Here, we have multiplied the elements of the third row (R_3) by a constant K and added them to the corresponding elements of the first row (R_1).

Symbolically :-

We write this equation as

$$R_1 \to R_1 + KR_3$$

Now again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} Kc_1 & Kc_2 & Kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \Delta + 0$ (since R_1 and R_3 are same)

Hence $\boxed{\Delta_1 = \Delta}$

Note :-

1) If Δ_1 is the determinant obtained by applying

$$R_i \to kR_i \quad \text{or} \quad C_i \to kC_i$$

to the determinant Δ, then

$$\Delta_1 = k\Delta.$$

ii) If more than one operation like

$R_i \to R_i + kR_j$ is done in one ~~step~~, step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operation

Example:

Prove that.

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix} = a^3$$

⇒ Soln:

$$L.H.S = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$,

We have,

$$\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we have

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we have

$$2a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0\begin{vmatrix} a+b & a+b+c \\ 0 & a \end{vmatrix}$$

$$+ 0\begin{vmatrix} a+b & a+b+c \\ a & 2a+b \end{vmatrix}$$

$$= a\left(a^2 - 0\right) - 0 + 0$$

$$= a^3 = R.H.S \quad \text{proved}$$

☆ Area of TRIANGLE

In earlier class, we have studied that the area of a traingle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by expression.

Then,

$$\text{Area of } \Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note:

i) Since area is a positive quantity we always take the absolute value of the determinant.

ii) If Area is given, use both positive and negative value of the determinant for calculation.

iii) The area of the triangle formed by three ~~points~~ collinear points is zero.

Example :-

Find the area of the triangle whose vertices are $(3, 8)$, $(-4, 2)$ and $(5, 1)$.

⇒ The area of triangle is given by.

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1) + (-4)(1-8) + 5(8-2)\right]$$

$$= \frac{1}{2}\left[3 + 28 + 30\right]$$

$$= \frac{1}{2} \times 61$$

$$= \frac{61}{2}$$

☆ Minors and Cofactors

Definition of Minors :-

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Note :

Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order $n-1$.

Example :-

Find the minor of element 6 in the determinant.

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Sol:→ Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= 8 - 14$$
$$= -6$$

Definition of Cofactor.

Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$\boxed{A_{ij} = (-1)^{i+j} . M_{ij}}$, where M_{ij} is minor of a_{ij}.

Example:

Find minor and cofactors of all the element of the determinant.

$$\begin{vmatrix} 1 & -3 \\ 5 & 6 \end{vmatrix}$$

Sol:- Minor of the element a_{ij} is M_{ij}.

Then,

$a_{11} = 1$, so, $M_{ij} = M_{11} =$ Minor of $a_{11} = 6$.

$M_{12} =$ Minor of element $a_{12} = 5$

$M_{21} = -3$

$M_{22} = 1$

Now, cofactor of a_{ij} is A_{ij} so,

$A_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot 6 = 6$

$A_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 5 = -5$

$A_{21} = 3$

$A_{22} = 1$.

Note :-

It is denoted by C_{ij} or A_{ij}.

ADJOINT AND INVERSE OF A MATRIX.

Adjoint of a Matrix.

The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$, denoted by Adj A is defined as the transpose of the matrix. $[A_{ij}]_{n \times n}$.

where A_{ij} is the cofactor of element a_{ij}.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then.

matrix formed by, cofactors of each element is

$$C = \begin{bmatrix} A_{11} & A_{22} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where $A_{11}, A_{12}, A_{13}, \ldots$ are cofactors of elements $a_{11}, a_{12}, a_{13}, \ldots$ respectively.

AND.

$$\text{Adj } A = C^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example :-

Find adj A For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol :→ $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$A_{11} = 4$, $A_{12} = -1$, $A_{21} = -3$, $A_{22} = 2$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$ Ans

Theorem 1 :-

If A be any given square matrix of order n, then

$$A(\text{adj } A) = (\text{adj } A)A = |A| I$$

where I is the identity matrix.

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}.$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A| I$$

$$\therefore A(\text{adj } A) = (\text{adj } A) A = |A| I.$$

<u>Example</u> :-

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore L.H.S. = A(\text{adj } A)$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 - 3(-1) & -6+6 \\ 4-4 & -3+8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 8 - 3 = 5$$

$\therefore$ R.H.S. $= |A| I = 5I$ Proved.

★ Singular Matrix.

A square matrix A whose determinant is zero is called a singular matrix.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

$\therefore |A| = 8 - 8 = 0$

then the determinant A is zero.

$\therefore$ A is a singular matrix.

Non Singular Matrix :

A square matrix whose determinant is not zero is called a non singular matrix.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$$

Therefore A is a non-singular matrix.

Theorem 2 :-

If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Example :

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+10 & 3+2 \\ 0+20 & 9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 20 & 13 \end{bmatrix}$$ is a non singular matrix.

AND.

BA is a also a non-singular matrix.

Theorem 3 :

The determinant of the product of matrices is equal to product of their respective determinants, that $|AB| = |A||B|$, where A and B are square matrices of the same order.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2 & 3+4 \\ 15+2 & 9+8 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 19 & 17 \end{bmatrix}$$

$$L.H.S = |AB| = \begin{vmatrix} 7 & 7 \\ 19 & 17 \end{vmatrix}$$

$$= 7 \times 17 - 7 \times 19$$
$$= 7(-2)$$
$$= -14$$

$$R.H.S = |A| \cdot |B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= (4-6)(10-3) = (-2)(7)$$

$$= -14 = L.H.S$$

Proved.

Theorem 4 :-

A square matrix A is invertible if and only if A is non-singular matrix.

Proof :-

Let A be an invertible matrix of order n and I be the identity matrix of order n.

There exists a square matrix B of order n

$$\therefore AB = BA = I$$

We know that :

$$A(\text{adj } A) = (\text{adj } A)A = |A| I$$

Dividing by $|A|$, we have

$$A\left(\frac{\text{adj } A}{|A|}\right) = \left(\frac{\text{adj } A}{|A|}\right) = \frac{|A| I}{|A|}$$

or, $$A\left(\frac{\text{adj } A}{|A|}\right) = \left(\frac{\text{adj } A}{|A|}\right)A = I$$

Now, $$\left(\frac{\text{adj } A}{|A|}\right)A = I$$

Post multiplication of A^{-1},

$$\therefore \left(\frac{\text{adj } A}{|A|}\right)A \cdot A^{-1} = I A^{-1}$$

or, $$\left(\frac{\text{adj } A}{|A|}\right) I = A^{-1}$$

$$\boxed{\therefore A^{-1} = \frac{\text{adj } A}{|A|}}$$

Inverse of a matrix :-

Two non-singular matrices A and B are called inverse of each other iff $AB = BA = I$.

Inverse of matrix A is usually denoted by A^{-1}.

It is also called reciprocal of a matrix.

Then by definition, we get.

$$\boxed{\therefore AA^{-1} = A^{-1}A = I}$$

Example :-

If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that

$A.\,adj\,A = |A| I$, Also find A^{-1}.

Soln: $|A| = 1(16-9) - 3(4-3) + 3(3-4)$

$\Rightarrow 1 \neq 0$.

Now, $A_{11} = 7$, $A_{12} = -1$, $A_{12} = -1$,
$A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$,
$A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, and
$A_{33} = 1$.

$$\therefore \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now, $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| . I$$

Also, $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ Ans.

★ Applications of Determinants And Matrices

We shall discuss application of Determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

~~Contis~~ Consistant System :-

A system of equation is said to be consistant if its solution (one or more) exists.

Inconsistant system.

A system of equations is said to be inconsistent if its solution does not exist.

Solution of system of linear equations using inverse of a matrix.

Let us express the system of linear equations as matrix equations solve them using inverse of the coefficient matrix.

Consider the system of equations.

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then, the system of equations can be written as.

$$AX = B.$$

$$\text{or, } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case 1:-

If A is a nonsingular matrix, then its inverse exists.

Now

$$AX = B \quad \text{——— (1)}$$

Premultiplication of A^{-1} in eq^n (1), we have,

$$A^{-1}AX = A^{-1}B$$

$$\text{or, } IX = A^{-1}B$$

$$\text{or, } \boxed{x = A^{-1}B}$$

Now,
we find A^{-1} then solve the equation

Case II :

If A is a singular matrix then

$|A| = 0.$

In this case, we find (adj A) B.

If $(\text{adj } A) B \neq 0$, then the solution does not exist and the system of equations is called inconsistent.

If $(\text{adj } A) B = 0$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example :-

Solve the system of equations.

$$2x + 5y = 1$$
$$3x + 2y = 7.$$

Soln: Given that

$$2x + 5y = 1$$
$$3x + 2y = 7$$

The system of equations can be written as

$$AX = B$$

Where,

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now

$$|A| = 4 - 15$$
$$= -11 \neq 0$$

Therefore, A is non-singular matrix and it has unique solution.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{|A|} (\text{adj } A)$$

$$= -\frac{1}{11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}^T$$

$$= \frac{-1}{11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}^T$$

$$\text{or } A^{-1} = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Then

$$X = A^{-1}B$$

$$\text{or } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} 2-35 \\ -3+14 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\left.\begin{aligned} \therefore\ x &= 3 \\ y &= -1 \end{aligned}\right\} \underline{\text{Ans.}}$$

Cramer's Rule :-

Consider the system linear equations :

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

We find

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now we find Δx which is obtained by suppressing the column of coefficient of x and replacing it by the column of constant term d_1, d_2 and d_3 on right hand side.

$$\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta}, \quad z = \frac{\Delta z}{\Delta}$$

where $\Delta \neq 0$.

Q Solve the following by using cramer's rule.

$$x - 2y + 3z = 2$$
$$2x - 3z = 3$$
$$x + y + z = 6$$

$\Rightarrow$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(0+3) + 2(2+3) + 3(2-0)$$
$$= 3 + 0 + 6$$
$$= 19$$

$$\Delta x = \begin{vmatrix} 2 & -2 & 3 \\ 3 & 0 & -3 \\ 6 & 1 & 1 \end{vmatrix}$$

$$= 2(0+3) + 2(3+18) + 3(3-6)$$
$$= 6 + 42 - 9 = 39.$$

$$\Delta y = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -3 \\ 1 & 6 & 1 \end{vmatrix}$$

$$= 1(3+18) - 2(2+3) + 3(18-3)$$
$$= 21 - 10 + 45$$
$$= 56$$

$$\Delta z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 6 \end{vmatrix}$$

$$= 1(0-3) + 2(18-3) + 2(2-0)$$
$$= -3 + 30 + 4 = 31$$

$$x = \frac{\Delta x}{\Delta} = \frac{39}{19}$$

$$y = \frac{\Delta y}{\Delta} = \frac{56}{19}$$

$$z = \frac{\Delta z}{\Delta} = \frac{31}{19}$$

DETERMINANTS

Determinant :-

To every matrix $A = [a_{ij}]$ of order n, we can associate a number (Real or complex) called determinant of the square matrix A, where

$$a_{ij} = (i, j)^{th} \text{ element of } A.$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of

A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Note:-

It is also denoted by $|A|$ or $\det(A)$ or Δ.

Note:-

(i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.

(ii) Only square matrix have determinant.

Determinant of a matrix of order One

Let $A = [a]$ be the matrix of order one, then determinant of A is defined to be equal to a.

Example:-

$$A = [2]$$

Soln:-

$$|A| = 2$$ Ans

Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2×2, then the ~~dert~~ determinant of A is defined as.

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Example:-

Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

Soluⁿ:-

$$\Delta = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= (2\times2) - (-1\times4)$$

$$= 4 - (-4)$$

$$= 4+4$$

$$= 8 \text{ Ans}$$

Determinant of a matrix of order 3x3

Determinant of a matrix of order three can be determinant by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column).

There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shows below

Consider the determinant of square matrix $A = [a_{ij}]_{3\times 3}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Raw (R_1)

$$\therefore |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example :-

Evaluate the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

Solu^n :-

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

Expanding along C_3

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-1-12) - 0(1-8) + 0(3+2)$$

$$= 4(-13) - 0 + 0$$

$$= -52 \quad \text{Ans}$$

PROPERTIES OF DETERMINANT

Property 1 :-

The value of the determinant remains unchanged if its Rows and columns are interchanged.

Varification :-

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

AND $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore \Delta = \Delta_1$

Note:- It follows from above property that if A is a square matrix, then

$\det(A) = \det(A')$ where A' is transpose of A.

Property 2 :-

If any two Rows (or columns) of a determinant are interchange, then sign of determinant changes.

verifacation:-

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

If interchanging first and third rows then new determinant obtained is given

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\boxed{\therefore \quad \Delta = -\Delta_1}$$

Property 3 :-

If any two rows (or columns) of a determinant are identical (all correspoing elements are same) then value of determinant is zero.

Example :-

Evaluate

$$\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$$

Solu^n:-

$$\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$$

Expanding along first row (R_1)

$$\Delta = 3\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= 3(6-6) - 2(6-9) + 3(4-6)$$

$$= 0 - 2(-3) + 3(-2)$$

$$= 6 - 6 = 0$$

Here R_1 and R_3 are identical.

Property 4 :-

If each element of a row (or column) of a determinant is multiplied by a constant K, then its value gets multiplied by K.

Verification :-

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_1 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix}$

AND Δ_1 be the determinant obtained by multiplying the elements of the first row by K.

then,

$$\Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta_1 = Ka_1(b_2c_3 - b_3c_2) - Kb_1(a_2c_3 - c_2a_3) + Kc_1(a_2b_3 - b_2a_3)$$
$$= K\left[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_2a_3)\right]$$
$$= K\Delta$$

Hence $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note :-

(i) By this property, we can take out ~~any~~ any common factor from any one row or any one column of a given determinant.

(ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (In the same ratio), then its value is zero.

Example :-

Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Soln :-

$$\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \cdot 0 = 0 \; (\text{If } R_1 = R_3).$$

Property 5 :-

If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more determinants)

Example :-

$$\begin{vmatrix} a_1 + d_1 & a_2 + d_2 & a_3 + d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 6 :-

If to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same.

i.e, the value of determinant remain same if we apply the operation

$$R_i \to R_i + K R_j \quad \underline{OR} \quad C_i \to C_i + K C_j$$

Verification :-

$$\text{Let } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{and } \Delta_1 = \begin{vmatrix} a_1 + K c_1 & a_2 + K c_2 & a_3 + K c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where Δ_1 is obtained by the operation

$$R_1 \to R_1 + K R_3$$

Here, we have multiplied the elements of the third row (R_3) by a constant K and added them to the corresponding elements of the first row (R_1).

Symbolically :-

We write this operation as

$$R_1 \to R_1 + KR_3.$$

Now again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} Kc_1 & Kc_2 & Kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + K \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \Delta + 0$ (Since R_1 and R_3 are same)

Hence $\boxed{\Delta_1 = \Delta}$

Note :-

(i) If Δ_1 is the determinant obtained by applying

$R_i \to K R_i$ or $C_i \to K C_i$ to

the determinant Δ, then

$$\Delta_1 = K\Delta.$$

(ii) If more than one operation like

$R_i \to R_i + K R_j$ is done in one step,

care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operation.

Example :-

Prove that

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix} = a^3.$$

Solu^n :-

$$L.H.S = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6a+3c \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have

$$\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying

$R_3 \to R_3 - 3R_2$, we have

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we have

$$= a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0\begin{vmatrix} a+b & a+b+c \\ 0 & a \end{vmatrix} + 0\begin{vmatrix} a+b & a+b+c \\ a & 2a+b \end{vmatrix}$$

$$= a(a^2 - 0) - 0 + 0$$

$$= a^3 = R.H.S$$ proved

AREA OF TRIANGLE

In ~~eatlier~~ earlier class, we have studied that the Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by expression.

Then

$$\text{Area of } \Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right],$$

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note :—

(i) Since area is a positive quantity we always take the absolute value of the determinant.

(ii) If Area is given, use both positive and negative value of the determinant for calculation.

(ii) The Area of the triangle formed by three collinear points is zero.

Example:-

Find the area of the triangle whose vertices are (3,8), (-4,2) and (5,1).

Soln:-

The are of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1) - 8(-4-5) + 1(-4-10)\right]$$

$$= \frac{1}{2}(3+72-14)$$

$$= \frac{61}{2} \text{ Ans}$$

MINORS AND COFACTORS

Definition of Minors:-

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. ~~Mo~~ Minor of an element a_{ij} is denoted by M_{ij}.

Note:-

Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order $n-1$.

Example:-

Find the minor of element 6 in the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Solu^n:-

Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= 8 - 14$$

$$= -6$$

Definition of cofactor :-

cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$\boxed{A_{ij} = (-1)^{i+j} \cdot M_{ij}}$$, where M_{ij} is minor of a_{ij}

Example :-

Find minor and cofactors of all the element of the determinant

$$\begin{vmatrix} 1 & -3 \\ 5 & 6 \end{vmatrix}$$

Solun :-

Minor of the element a_{ij} is M_{ij}.

then

$a_{11} = 1$, so $M_{ij} = M_{11} =$ minor of $a_{11} = 6$

$M_{12} =$ minor of element $a_{12} = 5$

$M_{21} = -3$

$M_{22} = 1$

Now, cofactor of a_{ij} is A_{ij} so,

$A_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot 6 = 6$

$A_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 5 = -5$

$A_{21} = 3$

$A_{22} = 1$

Note:-

It is denoted by C_{ij} or A_{ij}.

ADJOINT AND INVERSE OF A MATRIX

Adjoint of a Matrix:-

The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$, denoted by Adj$A$ is defined as the transpose of the matrix. $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of element a_{ij}.

suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

matrix formed by, cofactors of each element is

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where $A_{11}, A_{12}, A_{13}, \ldots$ are cofactors of elements $a_{11}, a_{12}, a_{13}, \ldots$ respectively

AND

$$\text{Adj } A = C^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example:-

Find adj A For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solun:-

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$A_{11} = 4, \; A_{12} = -1, \; A_{21} = -3, \; A_{22} = 2$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Ans

Theorem 1 :-

If A be any given square matrix of order n, then

$$A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I.$$

Where I is the identity matrix.

Proof :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\therefore A\,(\text{adj}\,A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A|\,I$$

$$\therefore A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I.$$

Example:-

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$$\text{adj} A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$\therefore$ L.H.S $= A(\text{adj} A)$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times4-3(-1) & -6+6 \\ 4-4 & -3+8 \end{bmatrix}$$

$$= \begin{bmatrix} 8-3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 8-3 = 5$$

$\therefore$ R.H.S $= |A| I = 5I$ proved

Singular Matrix :-

A square matrix A whose determinant is zero is called a singular matrix.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

$\therefore |A| = 8 - 8 = 0$

then the determinant A is zero.

$\therefore$ A is a singular matrix.

Non Singular Matrix :-

A square matrix whose determinant is not zero is called a non-singular matrix.

Example :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

Therefore A is a non singular matrix.

Theorem 2 :-

If A and B are non singular matrices of the same order, then AB and BA are also non singular matrices of the same order.

Example:- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+10 & 3+2 \\ 0+20 & 9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 20 & 13 \end{bmatrix}$$ is a non singular matrix.

AND

BA is a also non singular matrix.

Theorem 3 :-

The determinant of the product of matrices ~~of~~ is equal to product of their respective determinants, that $|AB| = |A|\,|B|$, where A and B are square matrices of the same order.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2 & 3+4 \\ 15+4 & 9+8 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 19 & 17 \end{bmatrix}$$

$$L.H.S \doteq |AB| = \begin{vmatrix} 7 & 7 \\ 19 & 17 \end{vmatrix}$$

$$= 7\times17 - 7\times19$$

$$= 7(-2)$$

$$= -14$$

$$R.H.S \doteq |A|.|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} . \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= (4-6)(10-3) = (-2)(7)$$

$$= -14 = R.H.S \text{ proved}$$

Theorem 4:-

A square matrix A is invertible if and only if A is nonsingular matrix.

Proof:- Let A be an invertible matrix of order n & I be ~~be~~ the identity matrix of order n.

There exists a square matrix B of order n

$\therefore AB = BA = I$

We know that

$A(\text{adj} A) = (\text{adj} A) A = |A| I$

Dividing by $|A|$, we have

$A\left(\frac{\text{adj} A}{|A|}\right) = \left(\frac{\text{adj} A}{|A|}\right) = \frac{|A| I}{|A|}$

or, $A\left(\frac{\text{adj} A}{|A|}\right) = \left(\frac{\text{adj} A}{|A|}\right) A = I$

Now, $\left(\frac{\text{adj} A}{|A|}\right) A = I$

Post multiplication of A^{-1}

$\therefore \left(\frac{\text{adj} A}{|A|}\right) A \cdot A^{-1} = I A^{-1}$

or, $\left(\frac{\text{adj } A}{|A|}\right) I = A^{-1}$

$$\boxed{\therefore \; A^{-1} = \frac{\text{adj } A}{|A|}}$$

Inverse of a Matrix :-

Two non singular matrices A and B are called inverse of each other iff $AB = BA = I$

Inverse of matrix A is usually denoted by A^{-1}.

It is also called reciprocal of a matrix.

Then by definition, we get

$$\boxed{\therefore \; AA^{-1} = A^{-1}A = I}$$

Example :- If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that

$A \cdot \text{adj} A = |A| I$, Also find A^{-1}.

Solutⁿ :-

$|A| = 1(16-9) - 3(4-3) + 3(3-4)$

$= 1 \neq 0$

Now, $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$

$A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$ and $A_{33} = 1$

$$\therefore \text{adj} A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now, $$A(\text{adj} A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I$$

Also $$A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ Ans

Statistics

classmate
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Definition.

It is the study of the collection analysis representation and organization of data.

Data :

eg:→ Market, Price list, Marksheet.

Representation of Data.

We can represent data by various way like :-

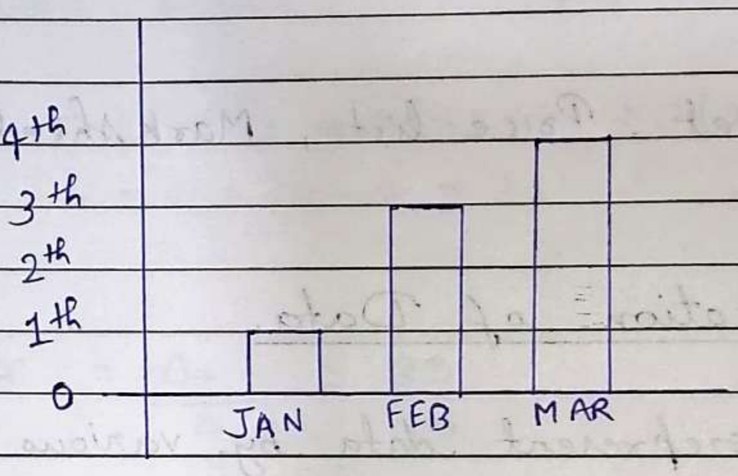
1. Table
2. Bar graph
3. Histograph
4. Frequency polygon
5. Pie Chart.

1 ~~Pie chart~~ Table.

x	f
8	3
15	5
18	7

2. Bar graph

Manufacture of medicine in companies

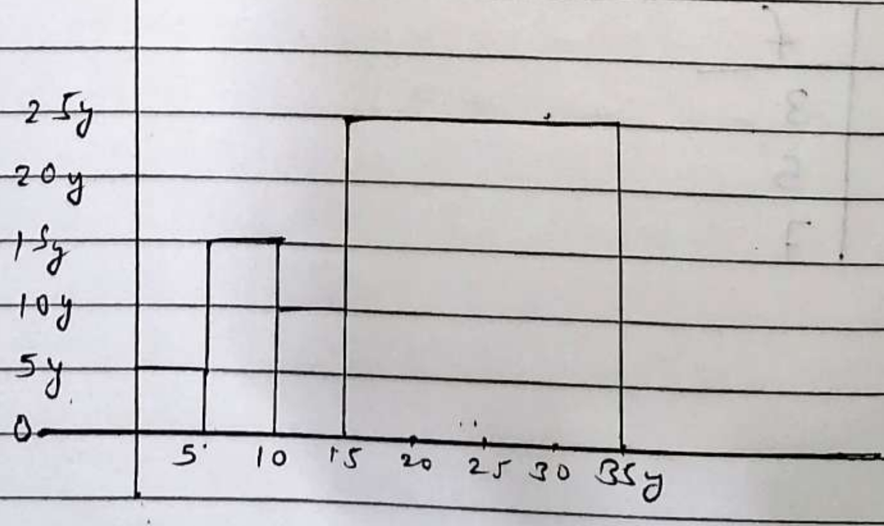


Note:

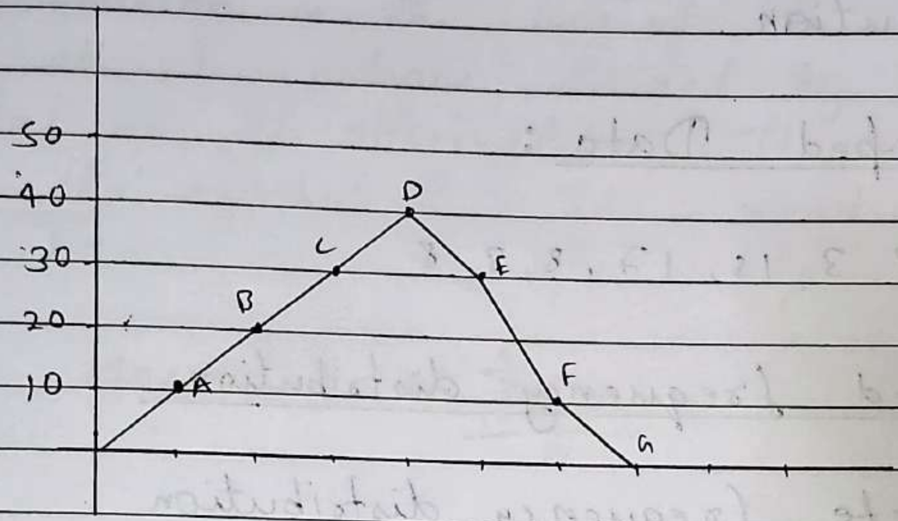
Breadth are same are called bar graph.

3. Histogram

Population in village.



4. Frequency polygon.

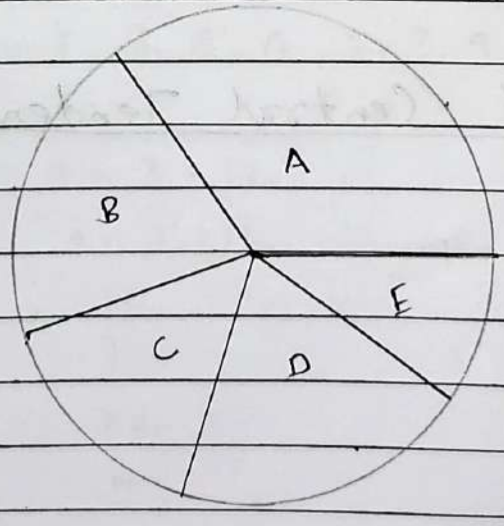


5. Pie Chart

Food items

A. A.) Orange - 35% → 126°
B.) Mango - 20% → 72°
C.) Grapes - 15% → 54°
D.) Apples - 20% → 72°
E) Extra - 10% → 36°

Degree = $\frac{\text{Value}}{\text{Total value}} \times 360°$



Types of frequency Distribution

There are two types of frequency distribution.

1. Ungrouped Data :

Ex :- 8, 3, 15, 17, 8, 3, 8

2. Grouped frequency distribution.

a) Discrete frequency distribution

eg.

x_i	F_i
8	18
3	5
10	15

b) Continuous frequency distribution

eg.

C.I.	F
0-5	1
5-10	4
10-15	6

Measures of Central Tendency.

1. Mean
2. Mode
3. Median

1. Mean

The mean (average) of a number of observation is the sum of the value of all observations divided by the total no. of observations. It is denoted by the symbol $\bar{x}$. It is read as x bar.

$$\text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} x_i}{N}$$

Ex:

Q. Marks score in exam by 5 students 40, 50, 75, 80, 95.

$$\text{Mean} = \frac{\text{Sum of total obs.}}{\text{No. of obs.}}$$

$$\frac{\sum_{i=1}^{n} x_i}{N} = \frac{40+50+75+80+95}{5} = \frac{390}{5} = 68$$

Q. 4, 5, 3, 8, 7, 3, 9, 4, 8, 3, 4, 5, 3, 8.

⇒

x_i	F_i	$F_i x_i$
3	4	$4 \times 3 = 12$
4	3	$4 \times 3 = 12$
5	2	10
7	1	7
8	3	24
9	1	9

$$\text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} F_i x_i}{\sum F_i}$$

$$\bar{x} = \frac{12+12+10+7+24+9}{4+3+2+1+3+1}$$

$$= \frac{74}{14} = 5.28 \text{ Ans}$$

20/02/22

Mode.

Mode of the group of observation is the variable that occurs most frequency in the distribution.

It is denoted by M_o.

Modal class.

The maximum frequency is called Modal class.

eg. The given observations are 8, 5, 3, 6, 3, 8, 9, 3, 5, 6, 3

$M_o = 4$ Ans.

$$\boxed{M_0 = l + \left(\frac{F_1 - F_0}{2F_1 - F_0 - F_2}\right) \times h}$$

Where,

l = lower limit of modal class.
h = Size of class interval.
F_1 = Frequency of modal class.
F_0 = Frequency of the class preceeding.
F_2 = Frequency of the class succeeding the modal class.

Q. A survey conducted on 20 households in a locality by a group of students ~~resel~~ resulted in the following frequency table of the number of family member in a household.

Family size	1 - 3	3 - 5	5 - 7	7 - 9	9 - 11
No. of families	7	8	2	2	1

Find the mode of this data.

⇒

F_i	x_i	$F_i x_i$
7	2	14
8	4	32
2	6	12
2	8	16
1	10	10
20		84

higher frequency = (3 - 5)

$$\text{Mean} = \frac{\sum F_i x_i}{\sum F_i} = \frac{84}{20} = 4.2$$

Then,

Modal Class $= (3-5)$

$h = 2$

$l = 3$

$F_1 = 8$

$F_0 = 7$

$F_2 = 2$

$$M_o = l + \left(\frac{F_1 - F_0}{2F_1 - F_0 - F_2}\right) \times h$$

$$= 3 + \left(\frac{8-7}{16-7-2}\right) \times 2$$

$$\Rightarrow 3 + \frac{1}{7} \times 2$$

$$= \frac{23}{7} = \underline{\underline{3.28}}$$

Median.

Median of group of observations in the value of observation is divides the group into two equal parts.

1) Data:

Arrange in ascending order.

ii) Find the number of observation.
If n is odd,

$$\text{median} = \frac{(n+1)}{2}^{th} \text{ observation}$$

Q. 12, 16, 12, 18, 38, 15, 36, 12, 14.

⇒ In ascending order the obs.

12, 12, 12, 14, 15, 16, 18, 36, 38

$n = 9.$

$$\text{Median} = \frac{(n+1)}{2}^{th} \text{ obs} = \left(\frac{9+1}{2}\right) \text{obs.}$$

$$= \left(\frac{10}{2}\right)^{th} \text{obs} = 5^{th} \text{ obs.}$$

$= 14$ Ans.

If n is Even.

$$\boxed{\text{Median} = \frac{\left(\frac{n}{2}\right)^{th} \text{ obs} + \left(\frac{n}{2}+1\right)^{th} \text{ obs}}{2}}$$

Q. 8, 3, 5, 13, 10, 9, 5, 7.

In ascending order

3, 5, 5, 7, 8, 9, 10, 13

$\therefore n = 8$

$$\text{Median} = \frac{\left(\frac{8}{2}\right)^{th} \text{ obs} + \left(\frac{8}{2}+1\right)^{th} \text{ obs}}{2}$$

$$= \frac{4^{th} \text{ obs} + 5^{th} \text{ obs}}{2}$$

$$= \frac{7+8}{2}$$

$$= \frac{15}{2}$$

$= \cancel{7}$ 7.5 Ans

If given question in class interval:

$$\text{Median} = l + \left(\frac{\frac{n}{2} - C.F.}{F}\right) \times h.$$

where,

The median class is just greater than or equal to cumulative frequency $\frac{N}{2}$.

where,

l = lower limit of median class.
n = number of observations
C.F. = Cumulative frequency of class preceeding the median class.

f = frequency of median class.
h = Class size (assuming class size to be equal).

Ex. 14.3

Q.

Monthly consumption (units)	x_i	No. of consumer	F_i	C.F.
65-85	75	4	300	4
85-105	95	5	475	9
105-125	115	13	1495	22
125-145	135	20	2700	42
145-165	155	14	2170	56
165-185	175	8	1400	64
185-205	195	4	780	68

$\frac{N}{2} = \frac{68}{2} = 34$

Median class > 34.

$\therefore$ Median class $= 125 - 145$

$l = 125,\ \frac{N}{2} = 34,\ CF = 22,\ f = 20,\ h = 20.$

$$\text{Median} = 125 + \left(\frac{34-22}{20}\right) 20.$$

$$= 125 + 12 = 137 \quad \underline{\text{Ans}}$$

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

$$= 125 + \left(\frac{20-13}{40-13-14}\right) \times 20$$

$$= 125 + \frac{7}{13} \times 20.$$

$$= 125 + 10.7$$

$$= \underline{\underline{135.7}}$$

Relation between three measures of central tendency (mean, mode, median).

$$\boxed{3\ \text{Median} = \text{Mode} + 2\,\text{Mean}}$$

Q.

C. I.	f_i	x_i	$f_i x_i$
0-10	5	5	25
10-20	3	15	45
20-30	6	25	150
30-40	4	35	140
40-50	1	45	45

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{405}{19} = 21.3 \quad \text{Ans}$$

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h.$$

$$= 20 + \left(\frac{6-3}{12-3-4}\right) \times 10.$$

$$= 20 + \frac{3}{5} \times 10$$

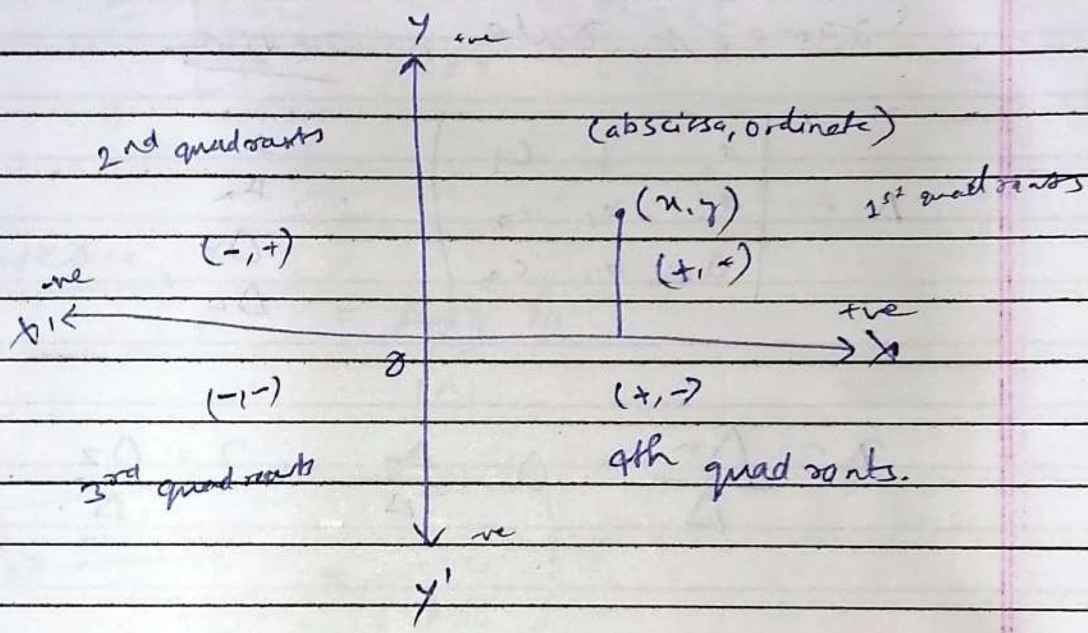
$$= 26 \quad \text{Ans}$$

$$3\ \text{Median} = \text{Mode} + 2\,\text{Mean}$$

$$\therefore \text{Median} = \frac{26 + (21.3) \times 2}{3} = \frac{26 + 42.6}{3} = 22.8$$

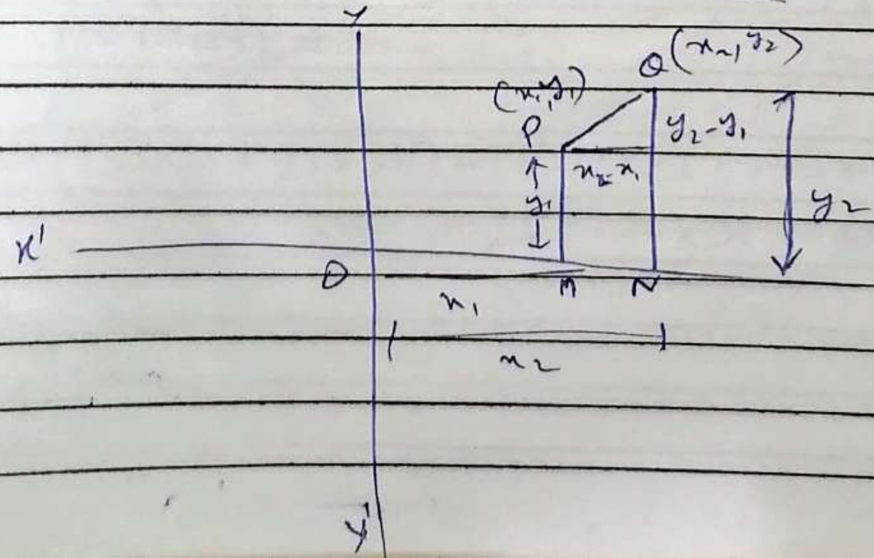
Matrix — St line
Determinant — Trigo
Cramer's rule — Calculus.
Statistics

Co-ordinate Geometry



There are four ~~qu~~ ~~coordinate~~ quadrants in plane figure

Distance between two points



$$h^2 = b^2 + p^2$$

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$$PQ = \sqrt{(PR)^2 + (QR)^2}$$

$$\therefore \boxed{PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

Example

If the points of AB

$A = (-5, 10)$

$B = (-8, 18)$

$$AB = \sqrt{(-8-(-5))^2 + (18-10)^2}$$

$$= \sqrt{(-3)^2 + (8)^2}$$

$$= \sqrt{9 + 64}$$

$$= \sqrt{73} \quad = \sqrt{73}$$

$= \cancel{5\sqrt{3}}$ Ans.

Area of triangle:

$$\text{Area of } \Delta = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

Q find the area of Δ, if points are $(5,6)$, $(3,9)$, $(-7,5)$

$$\text{Ar. of } \Delta = \frac{1}{2}\left[5(9-5)+3(5-6)+(-7)(6-9)\right]$$

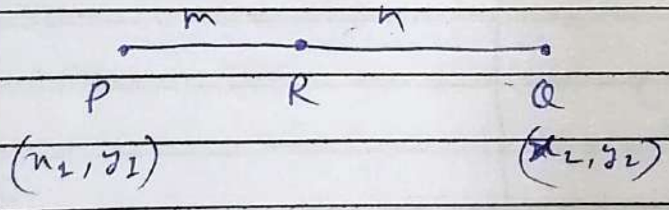
$$= \frac{1}{2}\left(5\times 4 + 3(-1) - 7(-3)\right]$$

$$= \frac{1}{2}\left(20 - 3 + 21\right)$$

$$= \frac{38}{2} = 19$$ ~~Ans~~ ~~eq~~. sq. units Ans

Section Formula:

1. Internal Divide



$$\frac{PR}{RQ} = \frac{m}{n}$$

Then x coordinate is of point R is equal to $\frac{mx_2 + nx_1}{m+n}$.

Then y coordinate of point R = $\frac{my_2 + ny_1}{m+n}$.

Example:

Find the coordinate of the point which divide the line segment joining the points (4, -3), (8, 5) in the ratio 3 : 1.

$$x = \frac{mx_2 + nx_1}{m+n}$$

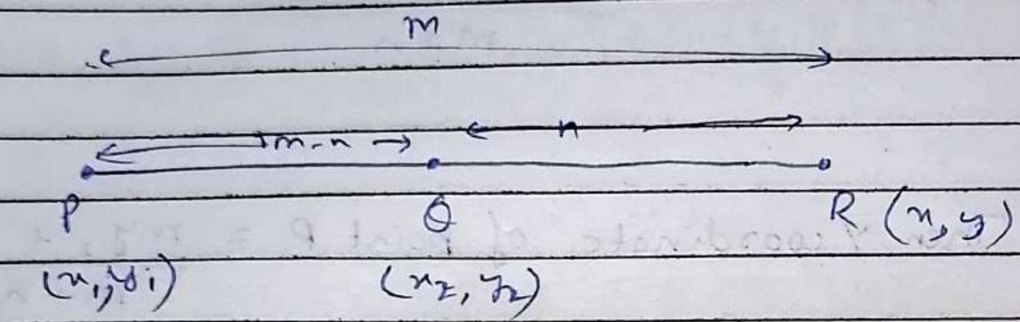
$$= \frac{3(8) + 1(4)}{3+1} = \frac{24+4}{4} = \frac{28}{4} = 7$$

$x = 7$ Ans

$$y = \frac{3(5) + 1(-3)}{4} = \frac{15-3}{4} = \frac{12}{4} = 3$$

$y = 3$. Ans

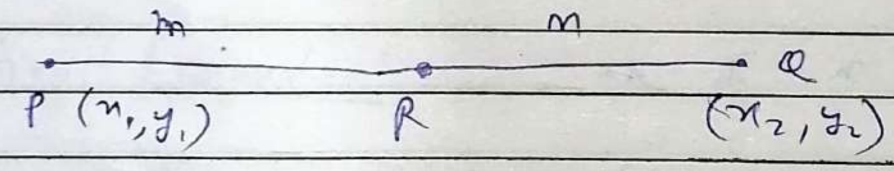
(ii) External Divide:



$$x = \frac{m x_2 - n x_1}{m - n}$$

$$y = \frac{m y_2 - n y_1}{m - n}$$

(iii) Mid-point OR Equal Divide.



$$x \rightarrow \frac{m x_2 + n x_1}{m + n} \Rightarrow \frac{m x_2 + m x_1}{m + m} = \frac{m(x_2 + x_1)}{2m}$$

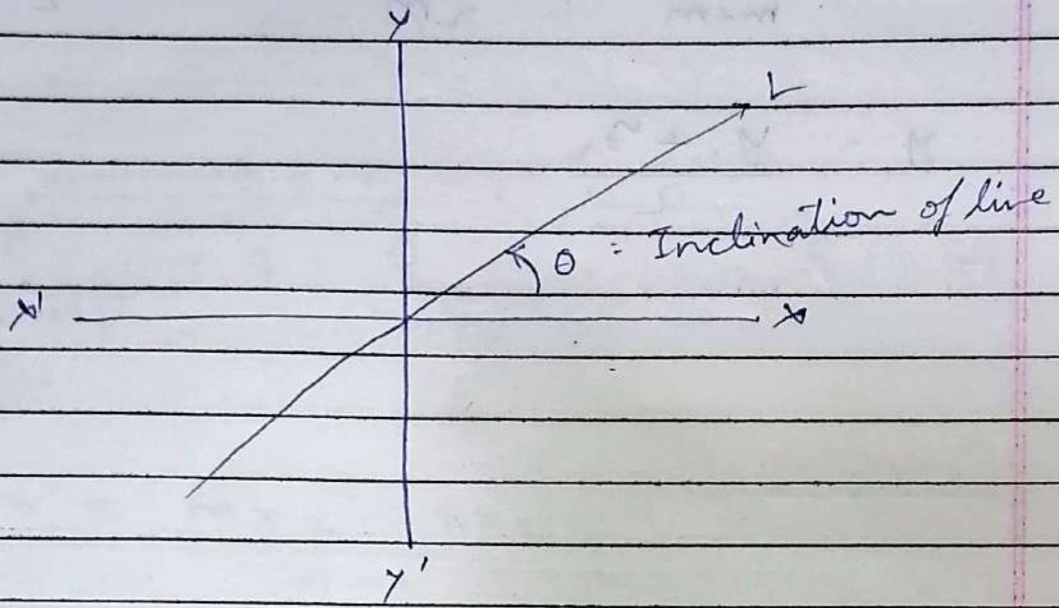
$$x = \frac{x_1 + x_2}{2}$$

$$y = \frac{my_2 + my_1}{m+m} = \frac{\cancel{m}(y_2 + y_1)}{2\cancel{m}} = \frac{y_2 + y_1}{2}$$

$$y = \frac{y_1 + y_2}{2}$$

Straight line

Slope of ~~line~~ line.



Definition: If θ is a ~~integration~~ inclinatio of a line L then $\tan\theta$ is called the slope / gradient of a line L. The slope of a line is denoted by "m" then slope of line ~~L is equal~~ L = m. i.e. $m = \tan\theta$.

$$\boxed{\text{Slope} = \tan\theta}$$

$$\boxed{m = \tan\theta}$$

$$\text{Slope} = \tan\theta$$
$$= \tan 30^\circ$$
$$= \frac{1}{\sqrt{3}}$$

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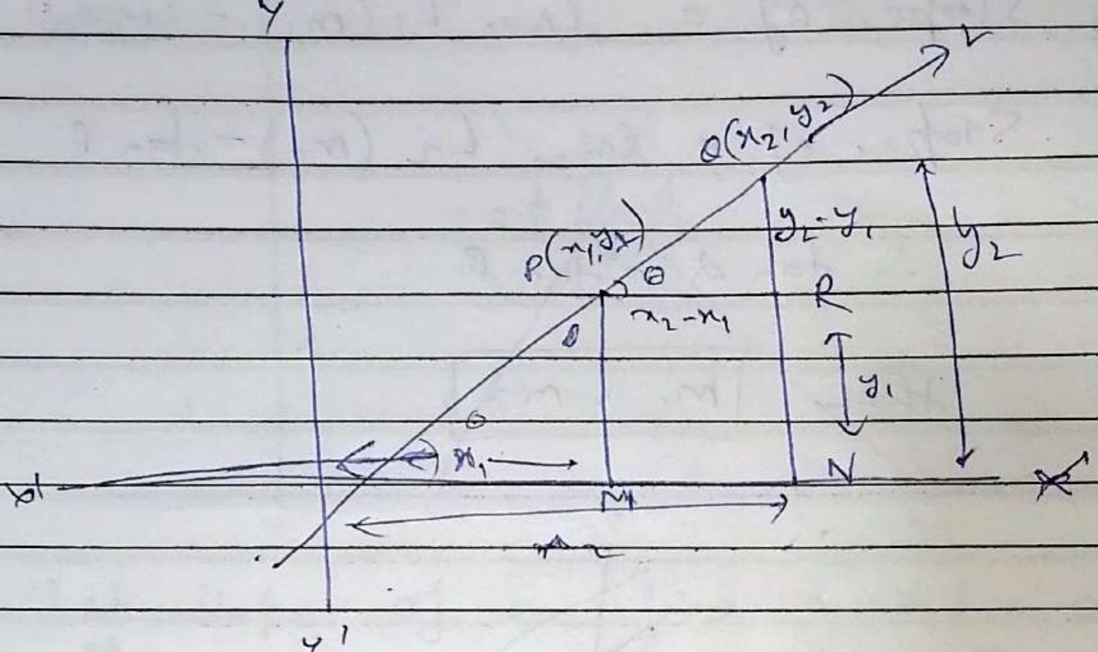
Slope of a line passing through two points.



In ΔPRQ

$M = \tan\theta$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

If a line passing through two point $P(x_1, y_1)$ and $Q(x_2, y_2)$ then slope of a line is $m = \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{-6}{4}$ (-1.5)

Q. Passing through the points (3, -2) and (-1, 4) then find the slope of a line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{(-1) - 3} = \frac{6}{-4} = \frac{-3}{2}$$

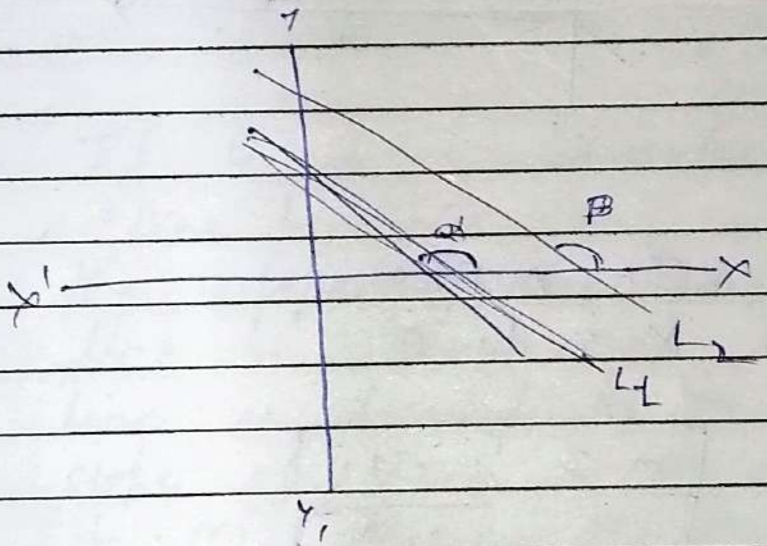
Condition of parallelism :-

Slope of a line $L_1 (m_1) = \tan \alpha$.

Slope of a line $L_2 (m_2) = \tan \beta$

$\therefore \tan \alpha = \tan \beta$

then $\boxed{m_1 = m_2}$



Conditionaly of perpendicularity

$\therefore \alpha = 90 + \beta$

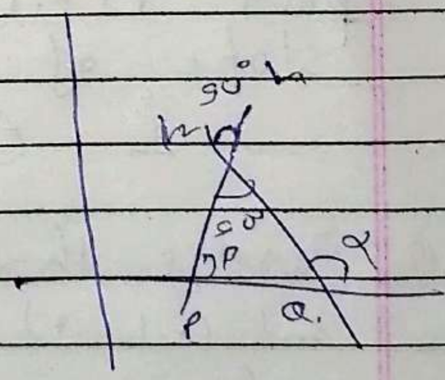
$\Rightarrow \tan \alpha = \tan(90° + \beta)$

$= \tan \alpha = -\cot \beta$

$= \tan \alpha = -\dfrac{1}{\tan \beta}$

$\boxed{\tan \alpha \cdot \tan \beta = -1}$

$\boxed{m_1 \cdot m_2 = -1}$



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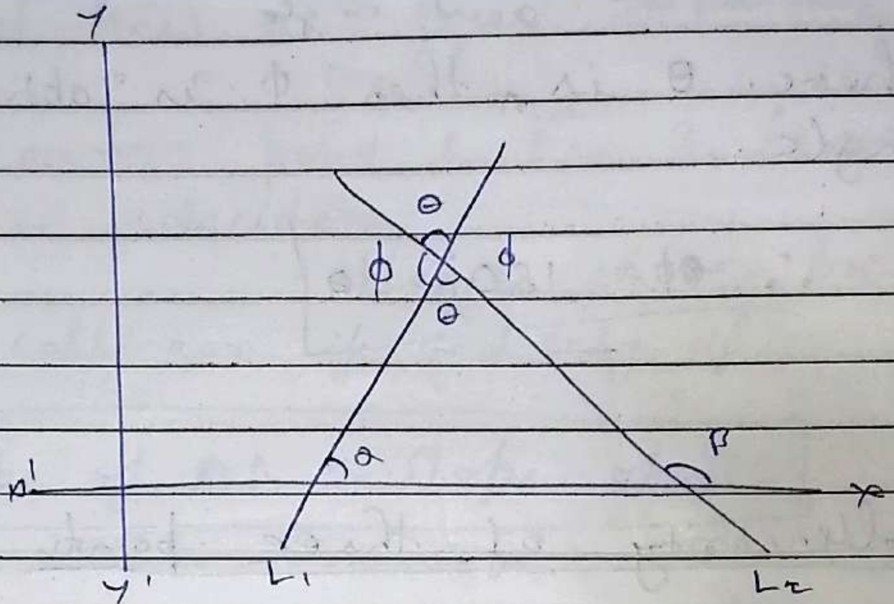
Angle between two lines



Let Slope of a line $L_1 = M_1$ and
" " " " $L_2 = M_2$.

We know that.

$\beta = \theta + \alpha$

$\Rightarrow \beta - \alpha = \theta$

$\Rightarrow \theta = \beta - \alpha$.

$\Rightarrow \tan\theta = \tan(\beta - \alpha)$

$\Rightarrow \tan\theta = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta \cdot \tan\alpha}$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 \cdot m_1}$$

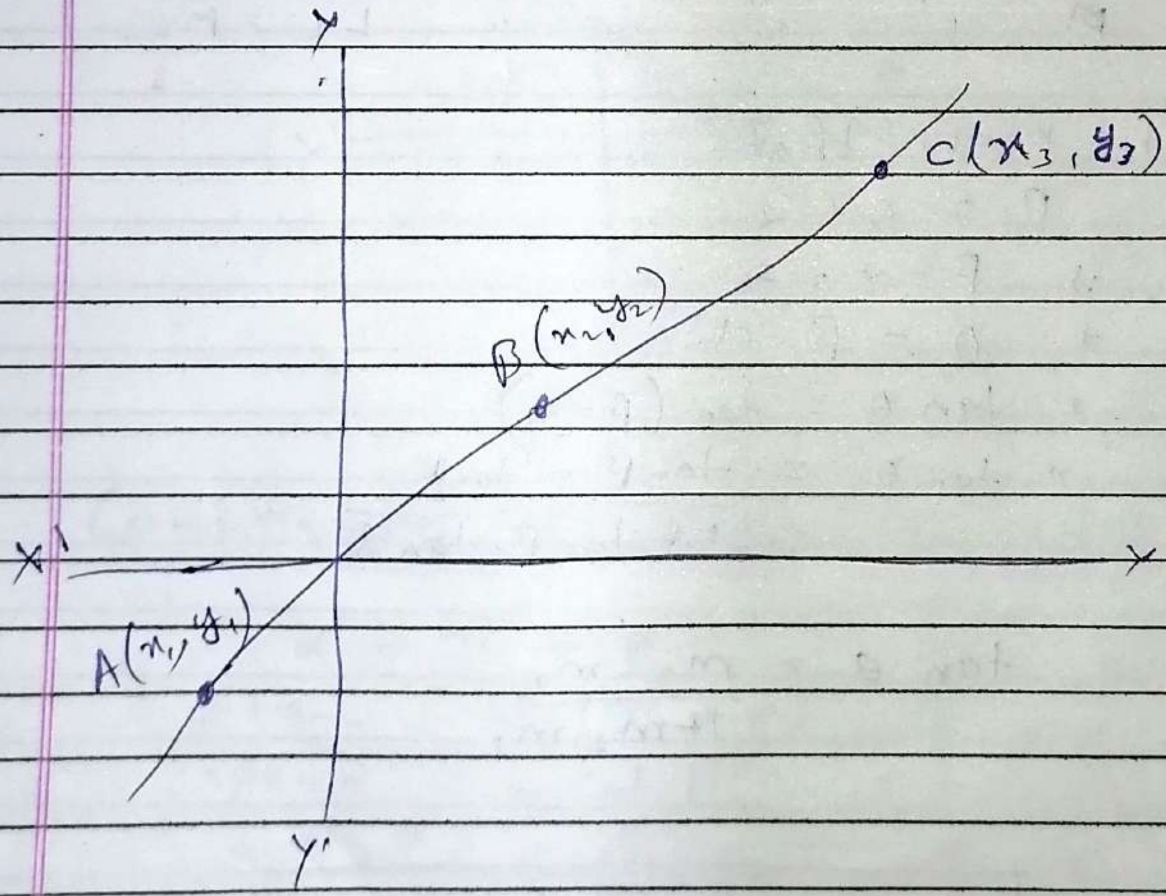
$$\boxed{\therefore \theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_2 \cdot m_1}\right)}$$

~~Collinearity~~ ~~θ~~

Where θ is acute angle then ϕ is obtuse angle.

$$\therefore \boxed{\phi = 180^\circ - \theta}$$

Collinearity of three points.



~~Slope of AB = Slope of BC~~

We know that slope of two parallel lines are equal, ~~∴ $m_1 = m_2$~~
$\therefore m_1 = m_2$

If common point B then 2 lines will be coincide.

Hence three points are collinear if and only if

$\boxed{\text{Slope of AB} = \text{Slope of BC}}$

Ex: 10.1

NCERT

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Q.8 $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ (Slope of AD = Slope of BC)

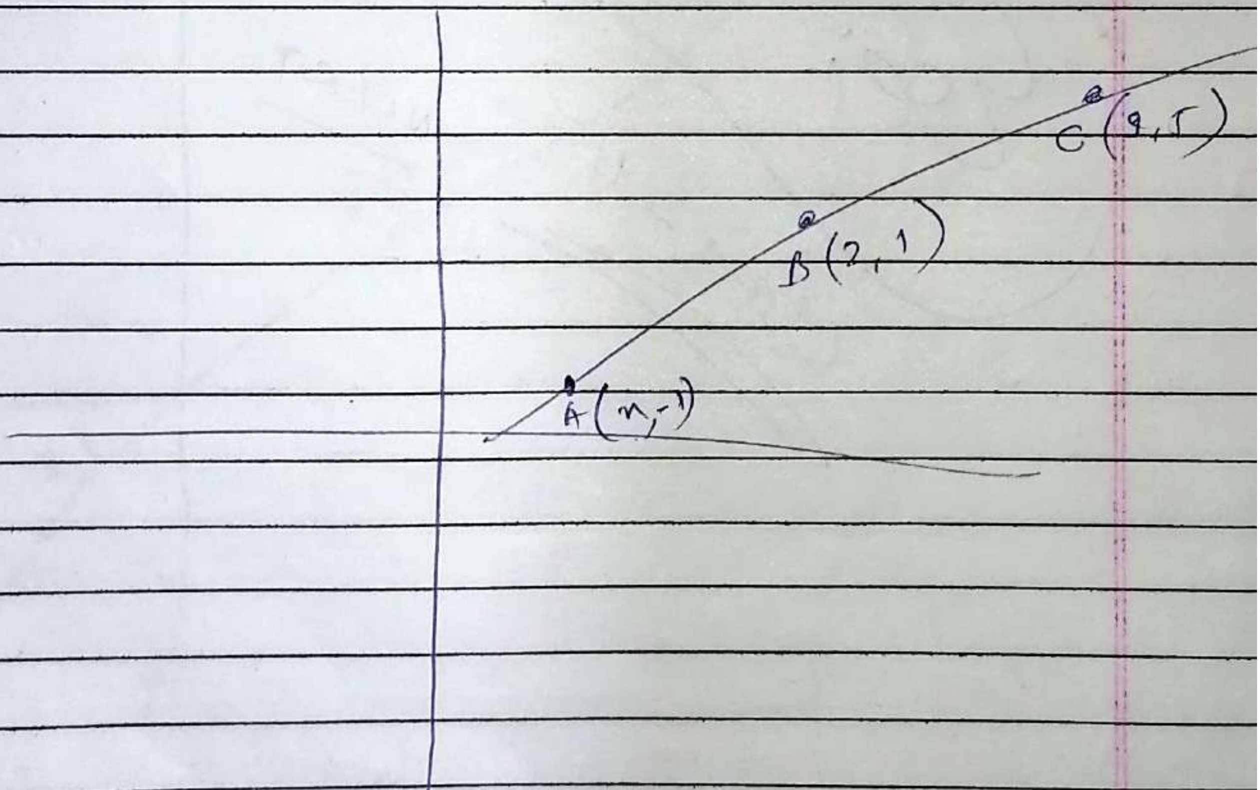
$= \frac{1+1}{2-x} = \frac{5-1}{4-2}$

$\frac{2}{2-x} = \frac{4}{2}$

$4 - 2x = 2$

$-2x = -2$

$x = 1$ Ans



(19) Slope of AB = Slope of BC

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{97 - 92}{1995 - 1985} = \frac{k - 97}{2010 - 1995}$$

$$\Rightarrow \frac{5}{10} = \frac{k - 97}{15}$$

$$= 75 = 10k - 970.$$

$$10k = 970 + 75$$

$$k = \frac{1045}{10}$$

$$k = 104.5$$

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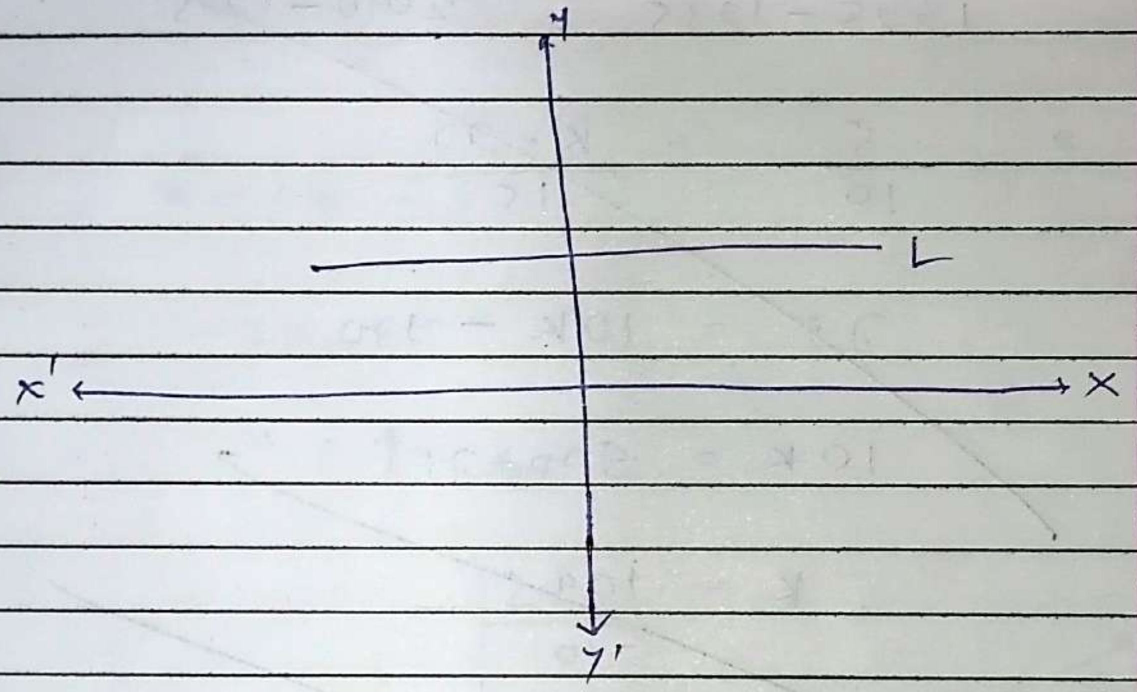
Various form of the equation of the line :

Horizontal line :

If a line parallel to x- axis then the line is horizontal.



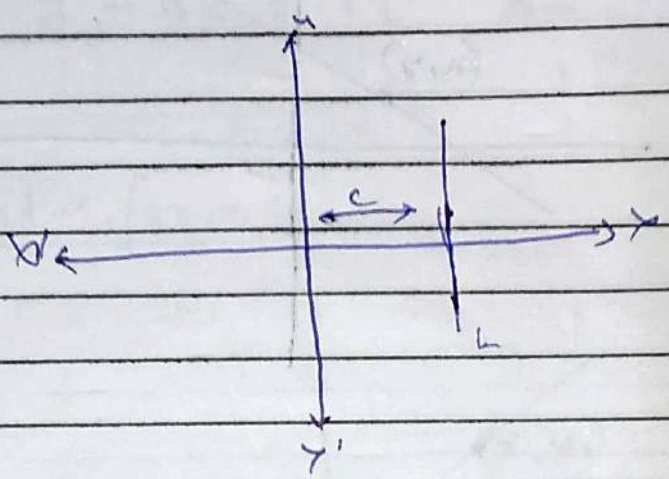
In a horizontal line, the distane of y-coordinate is fixed and x-coordinate always change.

$$\therefore \boxed{y = c}$$

The equation of x-axis is $\boxed{y = 0}$.

Verticle line:

If a line parallel to y-axis then the line is Vertical.

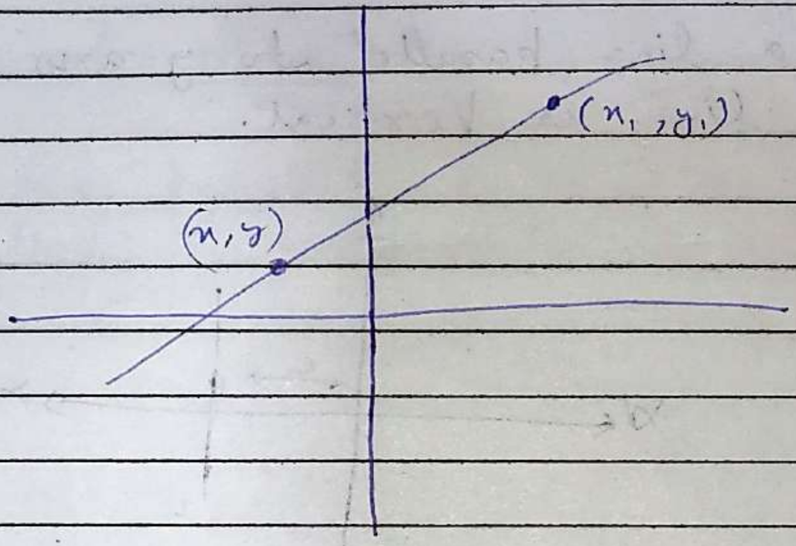


In a vertical line, the distance of x-~~axis~~ coordinate is fixed and y-coordinate always change.

$$\therefore \boxed{x = c}$$

The equation of y-axis is $\boxed{x = 0}$.

Point Slope form :-



Given the slope of line m and the point of a line $= (x_1, y_1)$.

we know that.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \boxed{y - y_1 = m(x - x_1)}$$

Q. Find the equation of a line passes through $(-2, 3)$ and with slope $= 4$.

$(x_1, y_1) = (-2, 3)$ $\quad m = 4$.

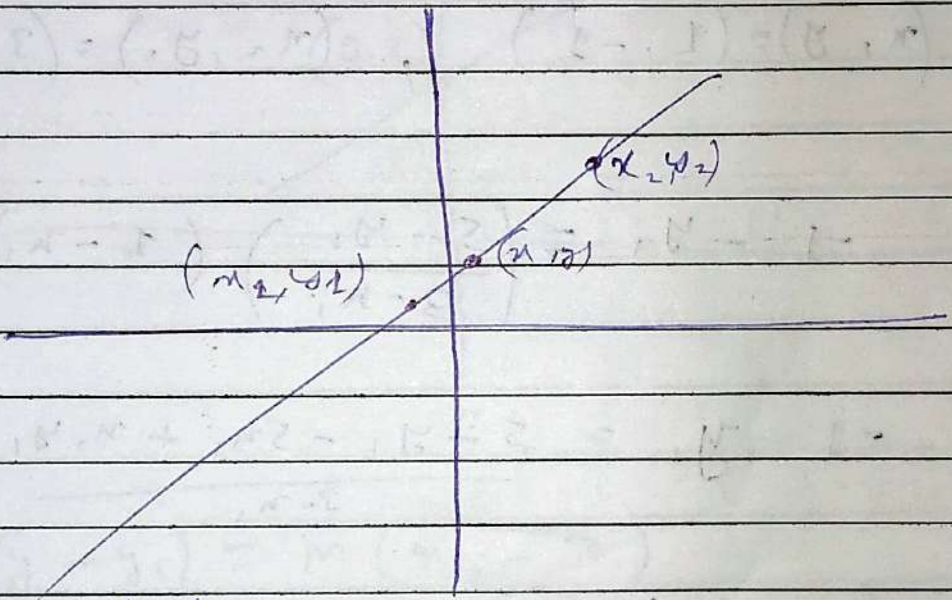
$$y - y_1 = m(x - x_1)$$

$$y - 3 = 4(x - (-2))$$

$$y - 3 = 4x + 8$$

$$\Rightarrow \quad y - 4x = 11 \quad \text{Ans}$$

Two point form:-



Slope of A and B.

Slope $\boxed{m = \dfrac{y_2 - y_1}{x_2 - x_1}}$

Slope of A and C.

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

Q. Write the equation of the line passing through the point (1, -1). and (3, 5).

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$(x, y) = (1, -1) \quad , \quad (x_2, y_2) = (3, 5)$$

$$-1 - y_1 = \left(\frac{5 - y_1}{3 - x_1}\right)(1 - x_1)$$

$$-1 - y_1 = \frac{5 - y_1 - 5x_1 + x_1 y_1}{3 - x_1}$$

$$(3 - x_1)(-1 - y_1) = 5 - y_1 - 5x_1 + x_1 y_1$$

$$-3 - 3y + x_1 + x_1 y_1 = 5 - y_1 - 5x_1 + x_1 y_1$$

$$-3 - 3y + x_1 + \cancel{x_1 y_1} - 5 + y_1 + 5x_1 - \cancel{x_1 y_1} = 0$$

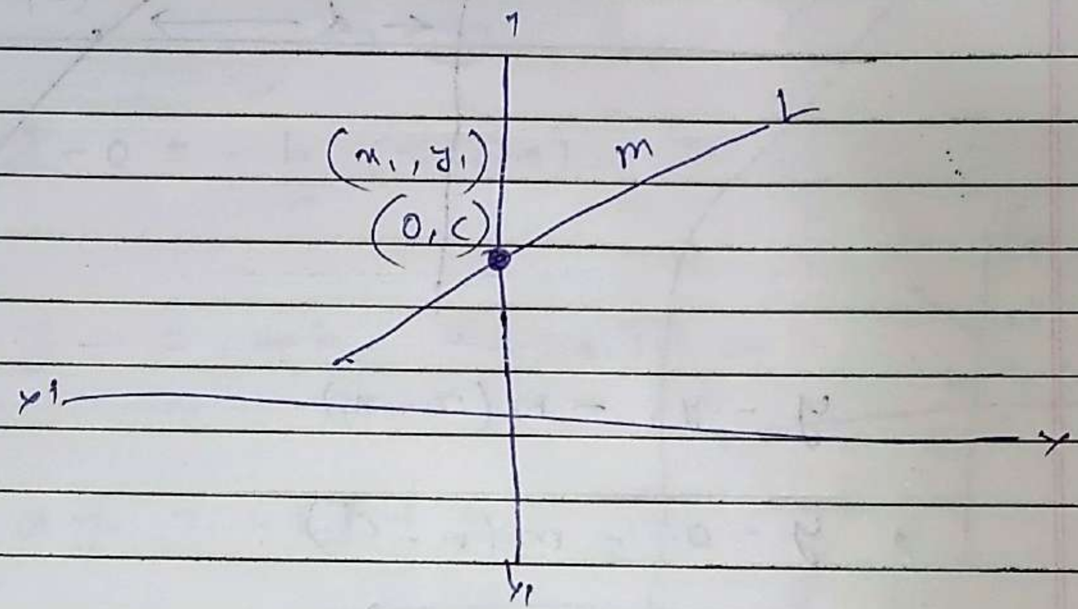
$$\boxed{-8 - 2y + 6x_1 = 0.}$$

$$\rightarrow \boxed{6x_1 - 2y - 8 = 0}$$ Ans

Slope Intercept form.

Case I:

Given y-intercept



$$(y - y_1) = m(x - x_1)$$

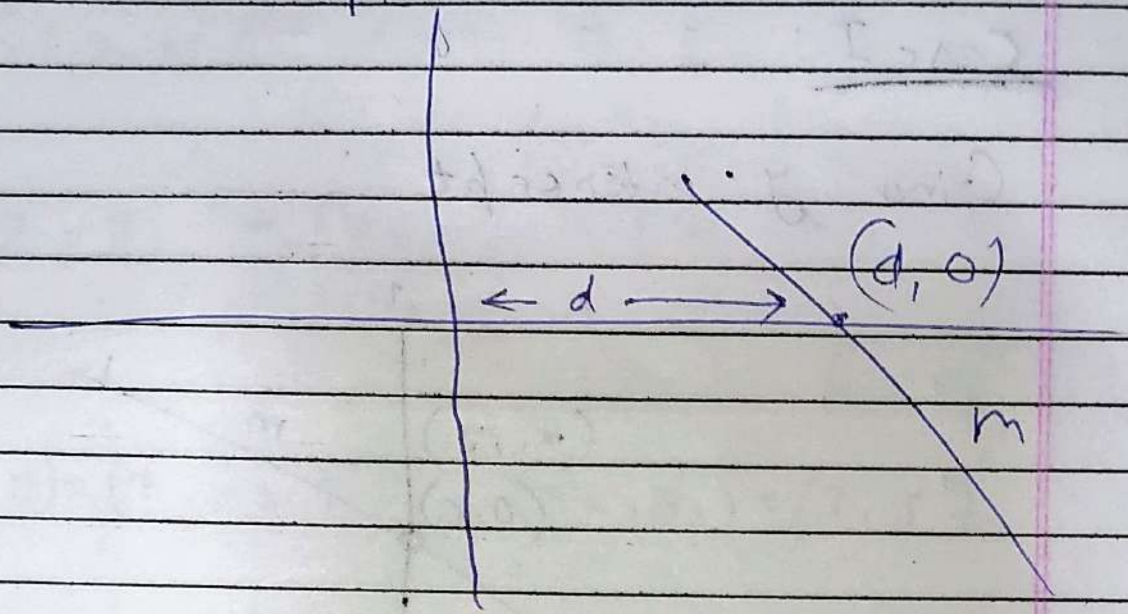
$$\Rightarrow y - c = m(x - 0)$$

$$\Rightarrow y - c = mx$$

$$\boxed{y = mx + c}$$

Case II

Given x - intercept



$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 0 = m(x - d)$$

$$\Rightarrow \boxed{y = m(x - d)}$$

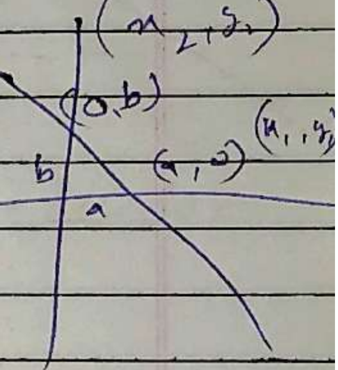
Intercept form :-

Slope of line $L = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b - 0}{0 - a} = \frac{-b}{a}$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = -\frac{b}{a}(x - a)$$

$$y = \frac{-b}{a}(x - a)$$



$$ay = -bx + ba$$

$$ay = -bx + ab$$

$$\Rightarrow bx + ay = ab$$

Dividing ab on both sides

$$\frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$$

$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Q. Find equation of the line which makes intercepts (−3 and 2) on the x and y-axis.

$$\frac{x}{a} + \frac{y}{b} = 1.$$

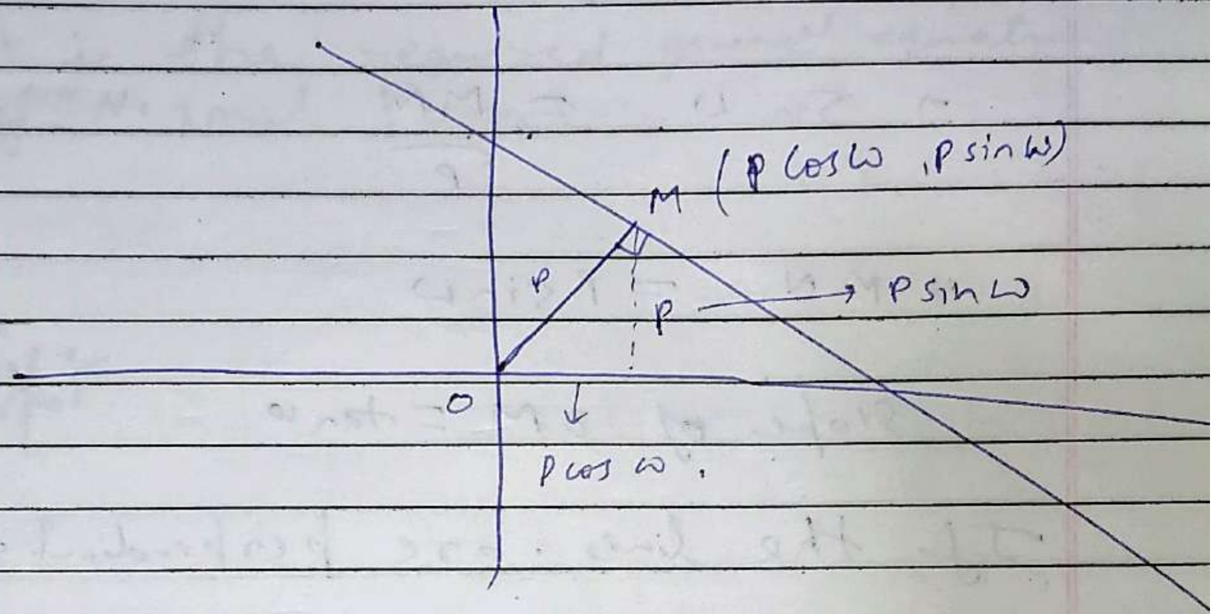
$$\frac{x}{-3} + \frac{y}{2} = 1.$$

$$\frac{-2x + 3y}{+6} = 1$$

$$\boxed{-2x + 3y = +6.}$$

$$\boxed{-2x + 3y - 6 = 0}$$ Ans.

Normal Form:



Given that the line of perpendicular (P) from origin to the line.

And angle ω is given.

In ΔOMN

$$\cos\omega = \frac{ON}{OM}$$

$$\cos\omega = \frac{ON}{P}$$

$$\Rightarrow ON = P\cos\omega$$

Again.

$$\sin\omega = \frac{MH}{OM}$$

$$\sin \omega = \frac{MN}{OM}$$

$$\Rightarrow \sin \omega = \frac{MN}{p}$$

$$\Rightarrow MN = p\sin\omega$$

Slope of $OM = \tan\omega$

If the lines are perpendicular

slope of $OM \times$ slope of $L = -1$

$\tan\omega \times$ slope of $L = -1$

$\Rightarrow$ slope of $L = \frac{-1}{\tan\omega} = \frac{-1}{\frac{\sin\omega}{\cos\omega}} = \frac{-\cos\omega}{\sin\omega}$.

$$y - y_1 = m(x - x_1)$$

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega)$$

$$y\sin\omega - p\sin^2\omega = -x\cos\omega + p\cos^2\omega$$

$$y\sin\omega + x\cos\omega = p\cos^2\omega + p\sin^2\omega$$

$$x\cos\omega + y\sin\omega = p(\cos^2\omega + \sin^2\omega)$$

$$\boxed{x\cos\omega + y\sin\omega = p}$$

It is the required general equation of Normal Form.

Ex:-

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Ex. 10.2

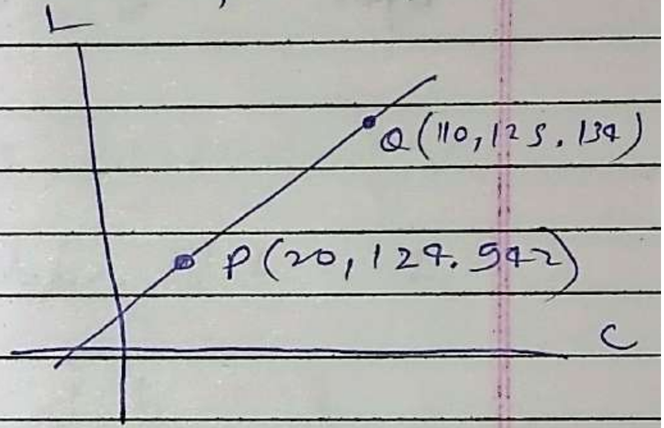
(16)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\frac{L = 124.942}{C = 20}\ l_1 \qquad \frac{L = 125.134}{C = 110}\ l_2$$

$\Rightarrow \cancel{L - 124.942}$

$= \cancel{125.134 - 124.942}$



$$\Rightarrow L - 124.942 = \frac{125.134 - 124.942}{110 - 20}(C - 20).$$

$$110L - 13743.62 - 20L + 2498.84$$

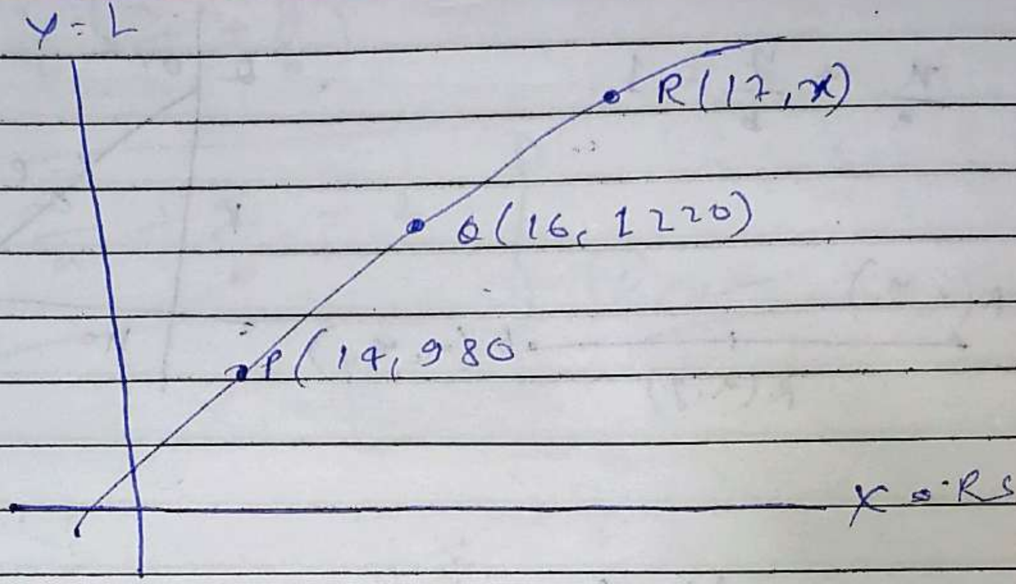
$$= 125.134C - 124.942C - 2502.68 + 13743.62.$$

$$90L - 13743.62 + 2498.84$$

$$= 0.192C - 2502.68 + 13743.62.$$

$$\boxed{90L - 0.192C - 22485.72 = 0}$$

(17)



Slope of PQ = Slope of QR.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{1220 - 980}{16 - 14} = \frac{x - 1220}{17 - 16}$$

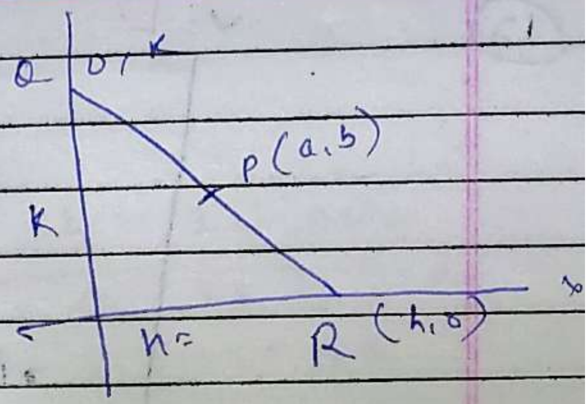
$$\Rightarrow \frac{240}{2} = \frac{x - 1220}{1}$$

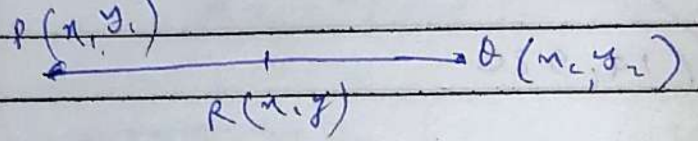
$$\Rightarrow x = 1220 - 120$$

$$x = 1340$$

(18)

$\frac{x}{a} + \frac{y}{b} = 1.$





x - coordinate $= \frac{x_1 + x_2}{2}$

$a = \frac{0 + h}{2}$

$\Rightarrow h = 2a$

y - coordinate $= \frac{y_1 + y_2}{2} = \frac{k + 0}{2}$,

$b = \frac{k}{2}$

$k = 2b.$

$\frac{x}{h} + \frac{y}{k} = 1$

$\frac{x}{2a} + \frac{y}{2b} = 1,$

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$$\frac{1}{2}\left(\frac{x}{a}+\frac{y}{b}\right)$$

$$\boxed{\frac{x}{a}+\frac{y}{b}=2.}$$ Proved.

$y = mx + c$

$y = m(x - d)$

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

$y - y_1 = m(x - x_1)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$x\cos w + y\sin\theta = p$

General Equation of a line.

We have studied general equation of 1^{st} degree in two variable.

$$\boxed{Ax + By + C = 0.}$$

Where A, B and C are constant. Such that A and B are not zero.

Graph of the equation $Ax + By + C = 0$ is always a straight line.

Therefore, any equation of the from $Ax + By + C = 0$ where A and B are not zero.

Simultaneously, that is called general equation or general linear eq^n.

Reduction of the general equation of a line to different standard form.

The general eq^n of a line can be reduced into various form of eq^n of a line.

1) Slope intercept form.

To reduce the eq^n $Ax + By + C = 0$ to the form $y = mx + c$. given the eq^n

$$Ax + By + C = 0$$

$$\Rightarrow By = -Ax - C$$

$$y = -\frac{A}{B}x - \frac{C}{B}$$

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

It is a slope of the eq^n.

Which is of the form $y = mx + c$.

Then slope $(m) = \left(-\frac{A}{B}\right)$ and $c = -\frac{C}{B}$

$$\text{Slope} \Rightarrow -\frac{\text{coefficient of } x}{\text{coefficient of } y}.$$

Q. Find the slope of the eqⁿ.

$$4x + 5y + 8 = 0.$$

$\Rightarrow$ $\text{Slope} = -\frac{4}{5}$ Ans.

(ii) Intercept form

To reduce equation $Ax + By + C = 0$.
to the ~~form~~ from $\frac{x}{a} + \frac{y}{b} = 1$.

This reduction is possible only when ~~C does not equal~~ $C \neq 0$.

Given the equation:-

$$Ax + By + C = 0.$$

$$\Rightarrow Ax + By = -C$$

$$\Rightarrow Ax + By = 1 \times (-C)$$

$$\Rightarrow \left(\frac{-A}{C}\right)x + \left(\frac{-B}{C}\right)y = 1.$$

$$\Rightarrow \boxed{\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1.}$$

Which is the form of $\frac{x}{a} + \frac{y}{b} = 1.$

where

$$\boxed{a = \frac{-c}{A}, \text{ and } b = \frac{-c}{B}}$$

Q. Equation of a line is $3x - 4y + 10 = 0$.
Find
(i) Slope
(ii) x and y intercept.

$$a = -\frac{10}{3}, \quad b = \frac{+10}{+4} = \frac{5}{2}$$

(i) Slope

Sol: $3x - 4y + 10 = 0$

$$-4y = -3x - 10$$
$$y = \frac{3}{4}x + \frac{5}{2}$$
slope.

ii) x and y Intercept:

Given the eqn

$$3x - 4y + 10 = 0$$

$$3x - 4y = -10$$

$$-\frac{3x}{10} + \frac{4}{10}y = 1.$$

$$\Rightarrow \frac{x}{\frac{-10}{3}} + \frac{y}{\frac{5}{2}} = 1.$$

$\therefore$ x-intercept is $= -\frac{10}{3}$,

y-intercept is $= \frac{5}{2}$.

iii) Normal form.

To reduce the eqn $Ax + By + c = 0$. to the form $x\cos\omega + y\sin\omega = p$.

$$Ax + By + c = 0.$$

$$\Rightarrow Ax + By = -c.$$

Put

$$\frac{A}{\cos\omega} = \frac{B}{\sin\omega} = -\frac{c}{p}.$$

We take

$$\frac{A}{\cos w} = \frac{C}{P}$$

$$\Rightarrow AP = -C \cdot \cos w.$$

$$\Rightarrow \cos w = \frac{-AP}{C}$$

And

$$\frac{B}{\sin w} = \frac{-C}{P}$$

$$\sin w = -\frac{BP}{C}.$$

Now

$$\sin^2 w + \cos^2 w = 1.$$

$$\Rightarrow \left(\frac{-BP}{C}\right)^2 + \left(\frac{-AP}{C}\right)^2 = 1.$$

$$\Rightarrow \frac{B^2P^2}{C^2} + \frac{A^2P^2}{C^2} = 1$$

$$\Rightarrow \frac{P^2}{C^2}(B^2 + A^2) = 1.$$

$$\Rightarrow P^2 \cdot (A^2 + B^2) = C^2$$

$$p^2 = \frac{C^2}{A^2+B^2}$$

$$\Rightarrow p = \sqrt{\frac{C^2}{A^2+B^2}}$$

$$p = \pm \frac{C}{\sqrt{A^2+B^2}}$$

Thus ~~Now~~ $\cos w = -\frac{Ap}{C}$

$$= -\frac{A}{C}\left(\pm \frac{C}{\sqrt{A^2+B^2}}\right)$$

~~Now~~

~~$\sin^2 w + \cos^2 w = 1$~~

$$\cos w = \pm \frac{A}{\sqrt{A^2+B^2}}$$

$$\sin w = \pm \frac{B}{\sqrt{A^2+B^2}}$$

$$\boxed{Ax + By + C = 0}$$

Intercept form → $\boxed{y = mx + c}$

$= \boxed{y = m(x - a)}$

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Normal Form :- $\boxed{x\cos\omega + y\sin\omega = p}$

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Reduce the General Eqn.

Ex:→

Q Reduce the eqn $\sqrt{3}x + y - 8 = 0$ into Normal form. Find the value of P and ω.

⇒ Given Eqn. is

$$\sqrt{3}x + y - 8 = 0.$$

$A = \sqrt{3}$, $B = 1$, $C = -8$.

$$P = \frac{C}{\sqrt{A^2 + B^2}} = \frac{-8}{\sqrt{(\sqrt{3})^2 + (1)^2}} = \frac{-8}{2} = -4.$$

Distance is always +ve.

$\therefore P = 4$

$$\cos \omega = \frac{\sqrt{3}}{2} = \frac{A}{\sqrt{A^2 + B^2}} = 30^\circ$$

$$\sin \omega = \frac{B}{\sqrt{A^2 + B^2}} = \frac{1}{2} = 30^\circ.$$

Obtuse angle $= 180^\circ - 30^\circ = 150^\circ$

Maths

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Q. Find the angle between the line $y - \sqrt{3}x - 5 = 0$. And $\sqrt{3}y - x + 6 = 0$

⇒ Given the eqn.

$$y - \sqrt{3}x - 5 = 0 \quad \text{---(I)}$$

And $\sqrt{3}y - x + 6 = 0$. ---(II)

Solving eqn (I), we have.

$$y - \sqrt{3}x - 5 = 0.$$

$$\Rightarrow y = \sqrt{3}x + 5$$

$$\therefore m_1 = \sqrt{3}$$

Solving eqn. (II)

$$\sqrt{3}y = x + 6$$

$$y = \frac{1}{\sqrt{3}}x - \frac{6}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}x - 2\sqrt{3}$$

$$\therefore m_2 = \frac{1}{\sqrt{3}}$$

Hence.

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \frac{1}{\sqrt{3}}.\sqrt{3}} = \frac{\frac{1-3}{\sqrt{3}}}{2}$$

$$= \frac{2}{\sqrt{3}} \times \frac{1}{2} = \frac{1}{\sqrt{3}} = 30^\circ \text{ Ans.}$$

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Q. Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2$ where b_1 and $b_2 \neq 0$, are.

Parallel

(i) If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

(ii) Perpendicular if $a_1a_2 + b_1b_2 = 0$.

(i) ⇒ Given the Eq^n :-

$$a_1x + b_1y + c_1 = 0 \quad \text{--- (I)}$$

and $$a_2x + b_2y + c_2 = 0 \quad \text{--- (II)}$$

From eq^n (I), we have.

$$a_1x + b_1y + c_1 = 0.$$

$$\Rightarrow b_1y = -a_1x - c_1$$

$$\Rightarrow y = \frac{-a_1x}{b_1} - \frac{c_1}{b_1}$$

$$\therefore m = -\frac{a_1}{b_1}$$

From Eqⁿ (ii), we have,

$$y = -\frac{a_2 x}{b_2} - \frac{c_2}{b_2}$$

$$\therefore m_2 = -\frac{a_2}{b_2}$$

If the lines are parallel.

$\therefore$ Slope of L_1 = Slope of L_2

$$\Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\Rightarrow \therefore \boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}}$$ Proved

(ii) $\therefore m_1 m_2 = -1,$

$$\Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1.$$

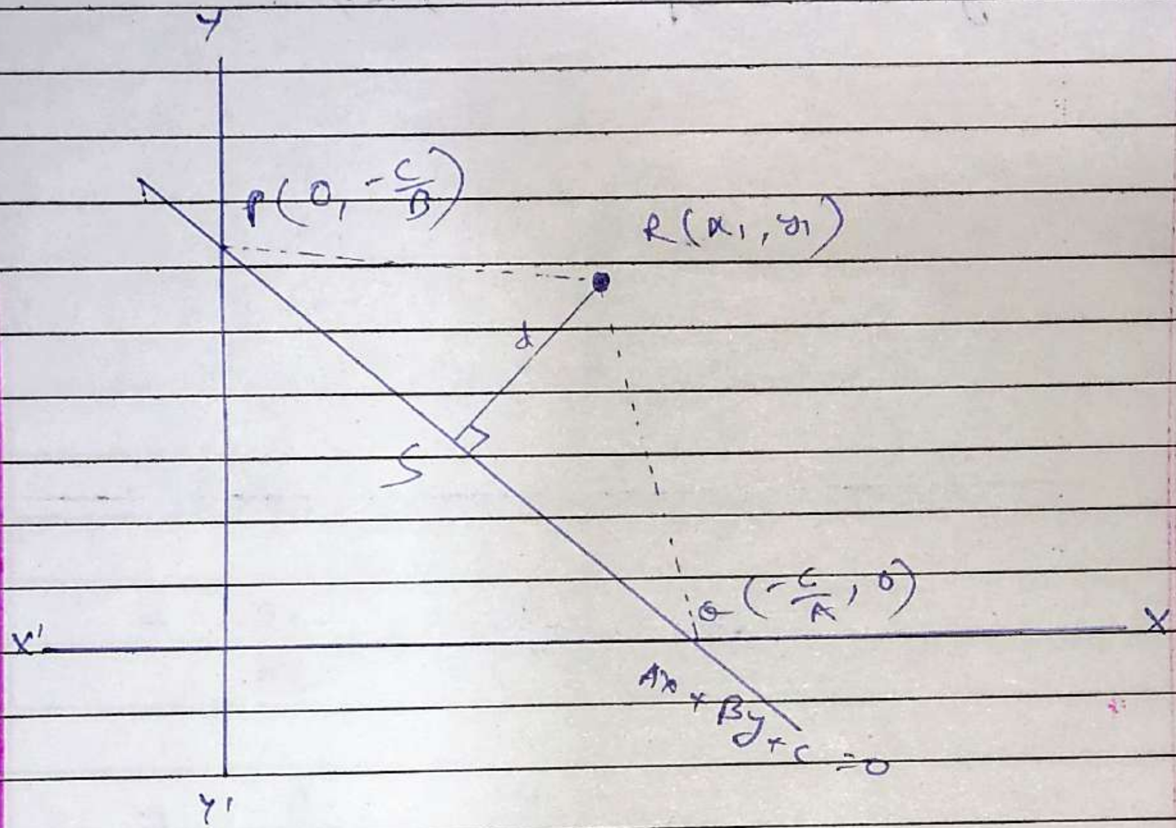
$$\Rightarrow \frac{a_1 a_2}{b_1 b_2} = -1.$$

$$\Rightarrow a_1 a_2 = -b_1 b_2$$

$$\Rightarrow \therefore \boxed{a_1 a_2 + b_1 b_2 = 0}$$ Proved

Q. Find the Eq^n of a line $\perp$ to the line $x - 2y + 3 = 0$ and passing through the point $(1, -2)$.

Distance of a ~~point~~ ~~line~~ point from a ~~point~~ line.



~~Here~~

~~PQ~~ = ~~d~~

The distance of a point from a line is the length of a perpendicular drawn from the point to the line. $Ax + By + C = 0$ whose distance from the point $R(x_1, y_1)$ is d."

The line meets the x and y - axis at the point ~~Q~~ ~~and~~ Q and P respectively.

$\therefore$ Points are

$$Q = \left(-\frac{C}{A}, 0\right)$$

$$P = \left(0, -\frac{C}{B}\right).$$

$$\text{Area of } \Delta = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

$$\Rightarrow \frac{1}{2} PQ.SR = \text{Area of } \Delta$$

$$\therefore SR = \frac{\text{Area of } \Delta}{\frac{1}{2} PQ.} \quad \text{--- (1)}$$

Here.

$$\text{Area of } \Delta = \frac{1}{2}\left| x_1\left(-\frac{C}{B} - 0\right) + 0(0 - y_1) + \left(-\frac{C}{A}\right)\left(y_1 + \frac{C}{B}\right) \right|$$

$$= \frac{1}{2}\left| -\frac{C}{B}x_1 - \frac{C}{A}y_1 - \frac{C^2}{AB} \right|$$

$$= \frac{1}{2}\left| -\frac{C(Ax_1 + By_1 + C)}{AB} \right|$$

$$= \frac{1}{2}\left|\frac{C}{AB}\right|(Ax_1 + By_1 + C).$$

$$PQ = \sqrt{\left(0 + \frac{C}{A}\right)^2 + \left(\frac{-C}{B} - 0\right)^2}$$

$$= \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B}}$$

$$= \left|\frac{C}{AB}\right| \sqrt{A^2 + B^2}$$

From (1).

$$SR(d) = \frac{\frac{1}{2}\left|\frac{C}{AB}\right|(Ax_1 + By_1 + C)}{\frac{1}{2}\left|\frac{C}{AB}\right|\sqrt{A^2+B^2}}$$

$$\boxed{d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}}$$

Proved.

Q. Find the distance of the point $(3, -5)$ from the line $3x - 4y - 26 = 0$.

$\Rightarrow$

$$d = \frac{3x - 4y - 26}{\sqrt{3^2 + (4)^2}}$$

$$d = \frac{3x - 4y - 26}{5}$$

$$x_1 = 3, \quad y_1 = -5$$

$$\therefore d = \frac{3(3) - 4(-5) - 26}{5}$$

$$= \frac{9 + 20 - 26}{5}$$

$$= \frac{3}{5} \text{ Ans.}$$

Distance between two parallel lines.

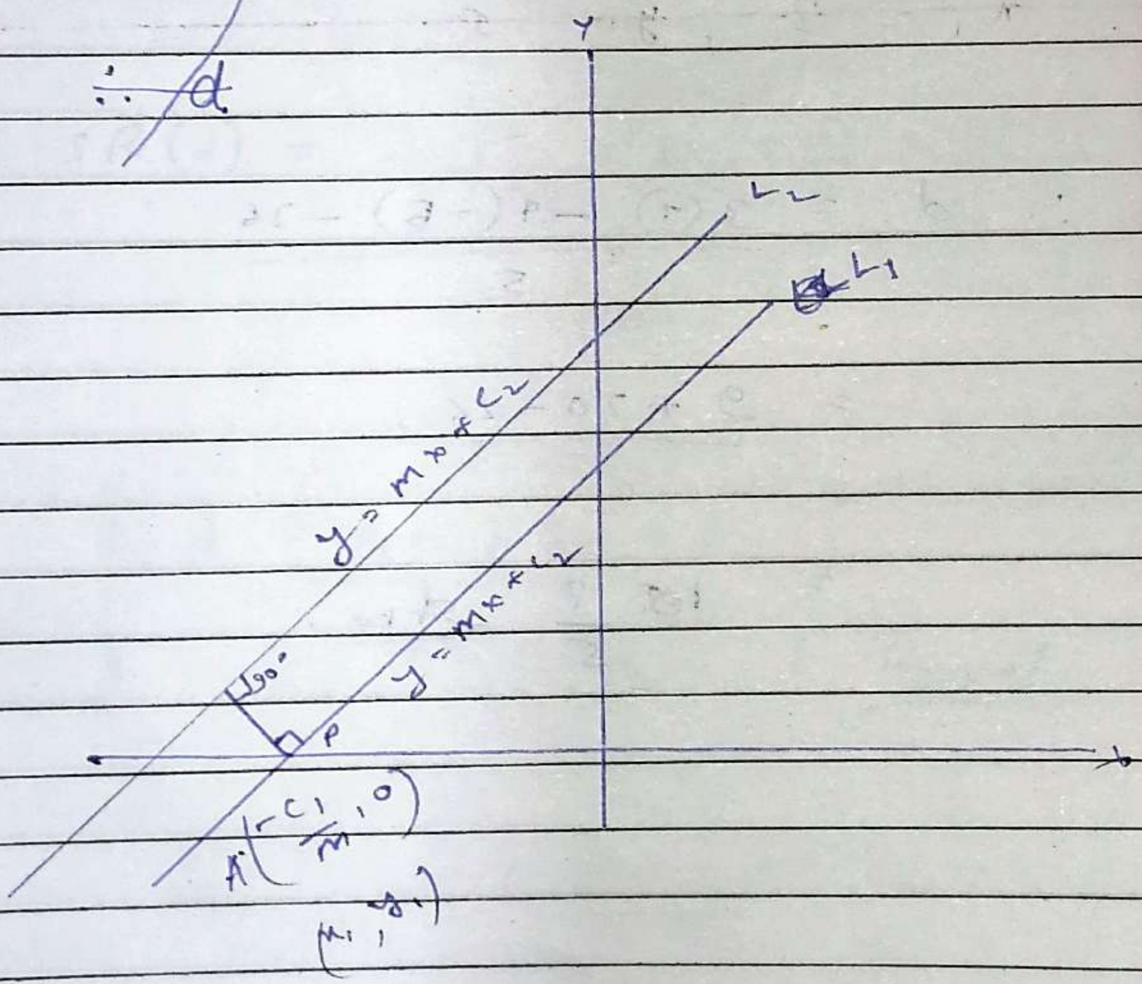
Given the eq^n of line L_2

$L_2 : y = mx + c.$

$\Rightarrow mx - y + c_2 = 0.$

$\therefore A = m, \quad B = -1, \quad C = c_2.$

$\therefore d$



We know that slope of two parallel lines are equal.

$$y = mx + c_1$$
$$y = mx + c_2$$

Line L_2 will ~~interest~~ intersect x-axis at the point $\left(-\frac{c_1}{m}, 0\right)$

Therefore distance between two line is equal to the length of the perpendicular from point A to L_2. Hence distance between the line L_1 and L_2

~~Given the~~

$$L_2 : y = mx + c_2$$

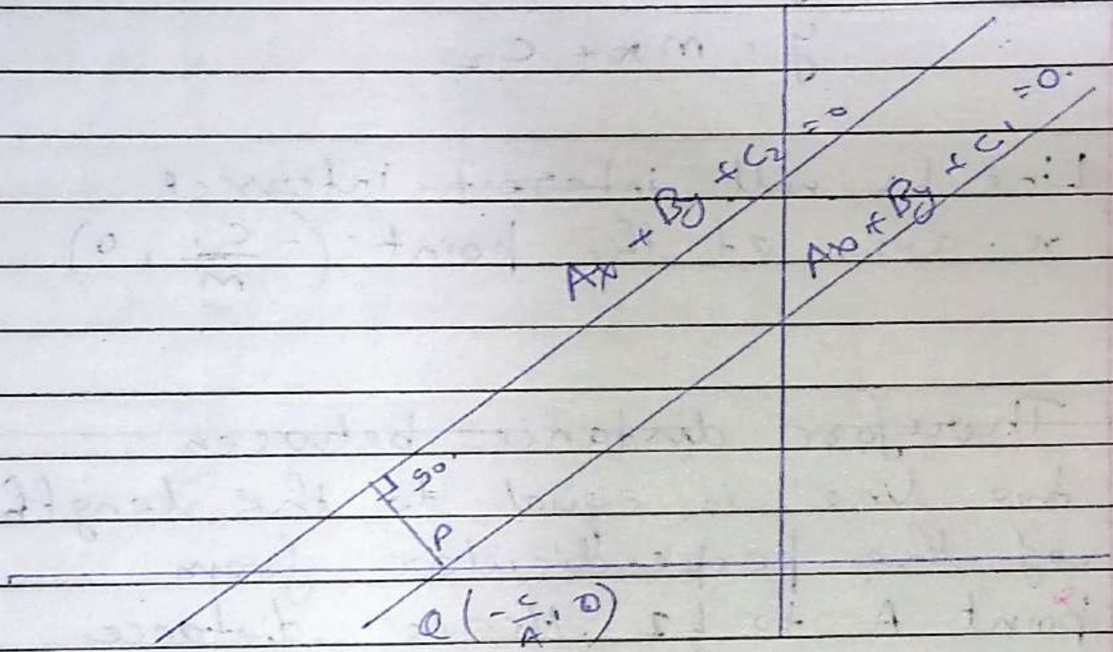
$$\Rightarrow mx - y + c_2 = 0$$

$$\therefore A = m,\ B = -1,\ C = c_2$$

$$\therefore \text{distance} = \left|\frac{Ax_1 + By_1 + c}{\sqrt{A^2 + B^2}}\right|$$

$$= \left|\frac{m \cdot \left(-\frac{c_1}{m}\right) + (-1)(0) + c_2}{\sqrt{m^2 + (-1)^2}}\right|$$

The



L_2 : $Ax + By + C_2 = 0$

And the point of line L_1

$$Q \equiv \left(-\frac{C}{A}, 0\right)$$

Distance =

Given the two lines L_1 and L_2 whose equations are $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ respectively.

The line L_1 passing through $Q\left(-\frac{C_1}{A}, 0\right)$

then,

$$\text{Distance} = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right|$$

$$= \left|\frac{A.\left(-\frac{C_1}{A}\right) + B.0 + C_2}{\sqrt{A^2 + B^2}}\right|$$

$$= \left|\frac{C_2 - C_1}{\sqrt{A^2 + B^2}}\right|$$ Proved.

Q. Find the distance between two parallel lines $3x - 4y + 7 = 0$ and $3x - 4y + 5 = 0$

$C_1 = 7$, $C_2 = 5$
$A = 3$, $B = 4$

$$D = \frac{C_2 - C_1}{\sqrt{A^2 + B^2}} = \frac{7 - 5}{\sqrt{9 + 16}}$$

$$= \frac{2}{5}$$ Ans.

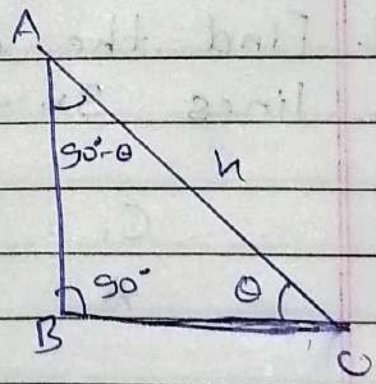
Trigonometry

Trigonometric Ratio of Angles.

Angle	0	30°	45°	60°	90°
Sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
Cosec A	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
Cot A	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Trigonometric Ratio of complementary Angle.

$\sin\theta = \frac{AB}{AC}$



$\cos\theta = \frac{BC}{AC}$

$\tan\theta = \frac{AB}{BC}$

$\cot\theta = \frac{BC}{AB}$, $\operatorname{cosec}\theta = \frac{AC}{AB}$, $\sec\theta = \frac{AC}{BC}$

$$\sin(90^\circ - \theta) = \frac{BC}{AC}$$

$$\cos(90^\circ - \theta) = \frac{AB}{AC}$$

$$\boxed{\frac{BC}{AC} = \frac{BC}{AC}}$$

$\sin(90^\circ - \theta) = \cos\theta$

$\cos = \sin$

$\tan = \cot$

$\cot = \tan$

$\operatorname{cosec} = \sec$

$\sec = \operatorname{cosec}$

Q. Prove that $\cos 38^\circ \cos 52^\circ - \sin 38^\circ . \sin 52^\circ = 0.$

$\Rightarrow$ L.H.S $= \cos(90^\circ - 52^\circ).\cos 52^\circ - \sin(90^\circ - 52^\circ).\sin 52^\circ$

$= \sin 52^\circ . \cos 52^\circ - \cos 52^\circ . \sin 52^\circ$

$= 0$ <u>proved</u>

Trigonometric Identities.

$$h^2 = p^2 + b^2$$

Dividing by h^2 on both side, we have.

$$\frac{h^2}{h^2} = \frac{p^2 + b^2}{h^2}$$

$$\cancel{\frac{h^2}{h^2}} = \frac{p^2}{h^2} + \frac{b^2}{h^2}$$

$$1 = \left(\frac{p}{h}\right)^2 + \left(\frac{b}{h}\right)^2$$

$$\Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}.$$

$\Rightarrow h^2 - p^2 = b^2$

$\sec^2\theta - \tan^2\theta = 1.$

$\sec^2\theta = 1 + \tan^2\theta$

$\sec\theta = \sqrt{1 + \tan^2\theta}$

$\tan^2\theta = \sec^2\theta - 1$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$\Rightarrow h^2 - b^2 = p^2$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$\operatorname{cosec}\theta = \sqrt{1 + \cot^2\theta}$

$\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$\cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$

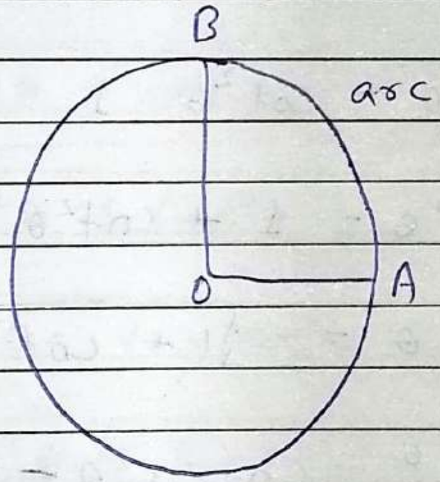
Measure of Angles

Radian or Circular measure.

Radian :

The angle subtended at the centre of circle by an arc whose length is equal to the radius of the circle is called radian.

It is denoted by 1^c.



If radius equal to 1 and arc = 1 then $\angle AOB$ is 1^c.

When θ is the angle in the radian subtended by an arc of length l at the centre of circle of radius are $\therefore \boxed{\theta = \frac{l}{r}}$

Where, l = arc of the circle
r = length of radius of circle.

$\boxed{l = \theta r}$, $\boxed{r = \frac{l}{\theta}}$

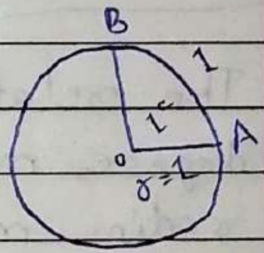
Ex:-

Q. If arc is 8 and radius = 4 then find the radian.

$$\theta = \frac{l}{r} = \frac{8}{4} = 2^c \text{ Ans}$$

Relation between degree and Radian

1 Revolution = $360°$.



$\Rightarrow 2\pi r = 360°$

$\Rightarrow 2\pi.1 = 360°$

$\boxed{\pi \text{ radian} = 180°}$

$\boxed{1 \text{ radian} = \frac{180°}{\pi}}$

$$\pi^c = 180^\circ$$

$$\pi^c = 180^\circ \times 1$$

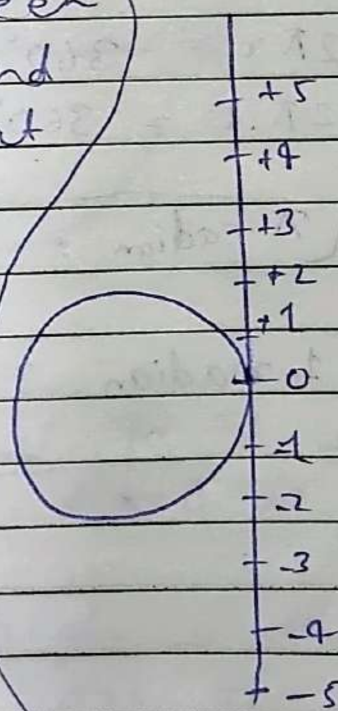
$$\cancel{\theta} = \frac{\cancel{180}\ \pi^c}{\cancel{\pi}\ 180} = 1$$

$$1^\circ = \left(\frac{\pi}{180}\right)^c$$

Relation between Radian and Real No.

→ Radian measure and Real no. are same

→ The relation between degree measure and radian measure at some common angle



Degree	0°	30°	45°	60°	90°	270°	360°
Radian	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/2$	2π

~~Ex. 3.1~~ Exercise 3.1

1..

⇒ (i) 25°

$$1^\circ = \left(\frac{\pi}{180}\right)^c$$

$$25^\circ = \frac{\pi}{180} \times 25$$

$$\Rightarrow \quad = \frac{\pi}{36}$$

$$= \left(\frac{5\pi}{36}\right)^c \quad \text{Ans}$$

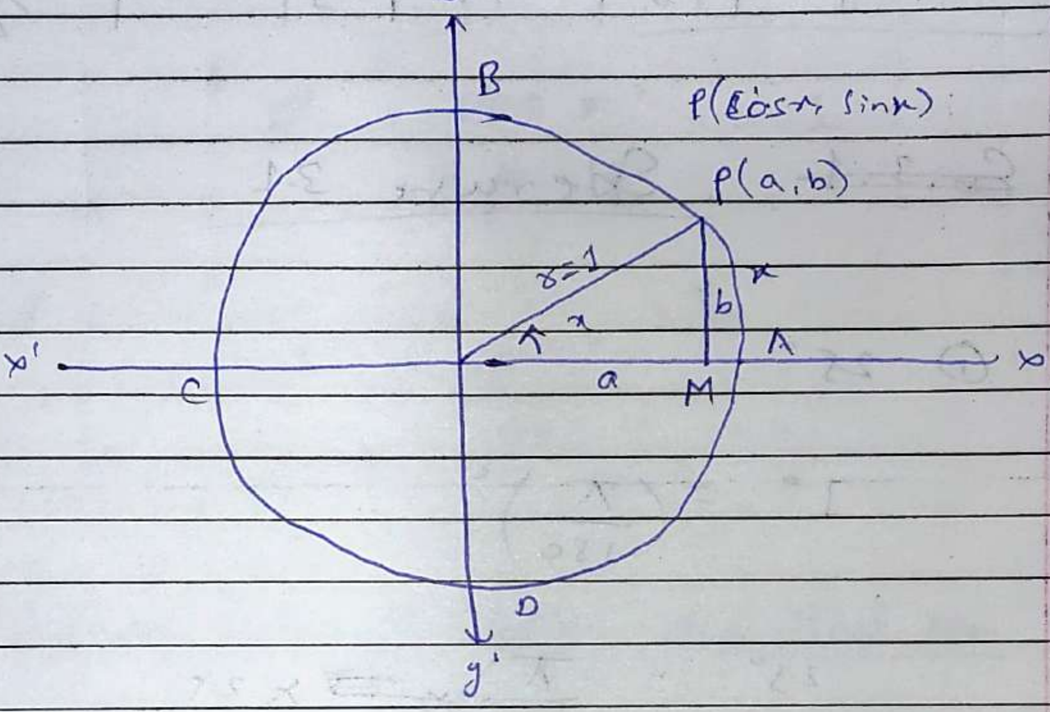
2. (i) $\frac{11}{16}^c$

$$1^c = \frac{180^\circ}{\pi}$$

$$\frac{11}{16} = \frac{180}{\pi} \times \frac{11}{16}$$

$$\Rightarrow 37^\circ \left(\frac{8}{9} \times 60\right)'$$

Trigonometric Functions



Let P(a, b) any point on the circle with angle and arc AP = x

Then angle $\angle AOP = x^c$

$OM = a$
$PM = b$
$OP = 1$ (Radius)

In $\triangle OPM$

$\sin x = \frac{p}{h}$

$\sin x = \frac{b}{1}$

$\Rightarrow b = \sin x$

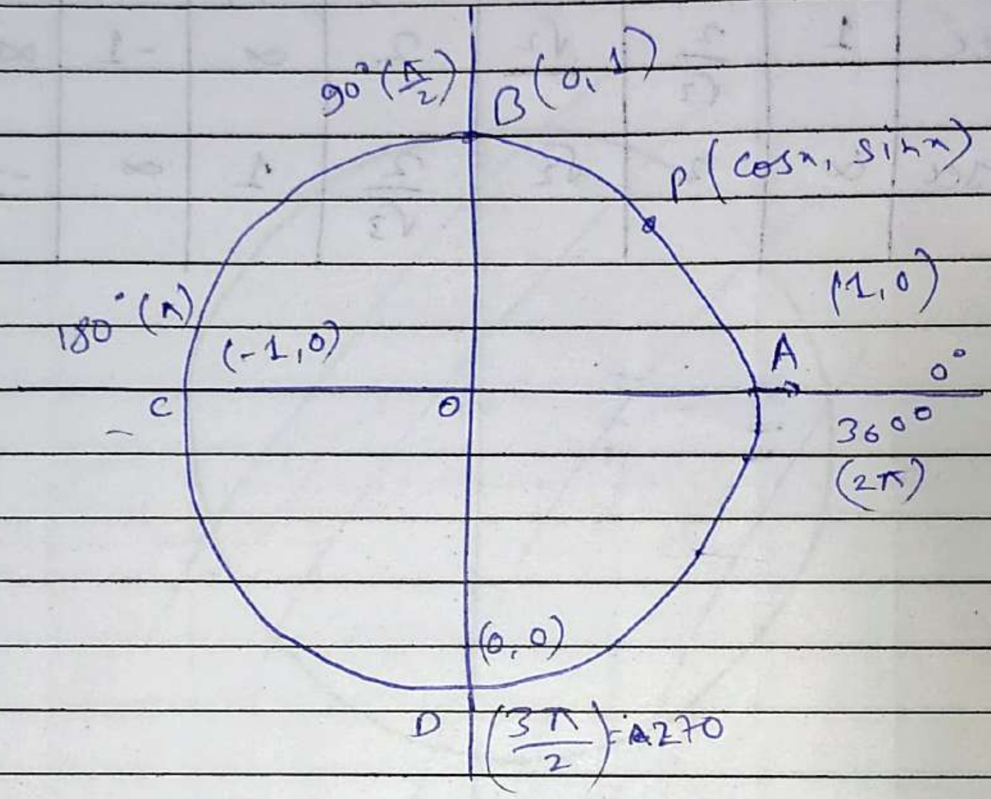
And,

$\cos x = \frac{b}{h}$

$= \frac{a}{1}$

$\Rightarrow a = \cos x$.

Therefore the point of P is (cos x, sin x).



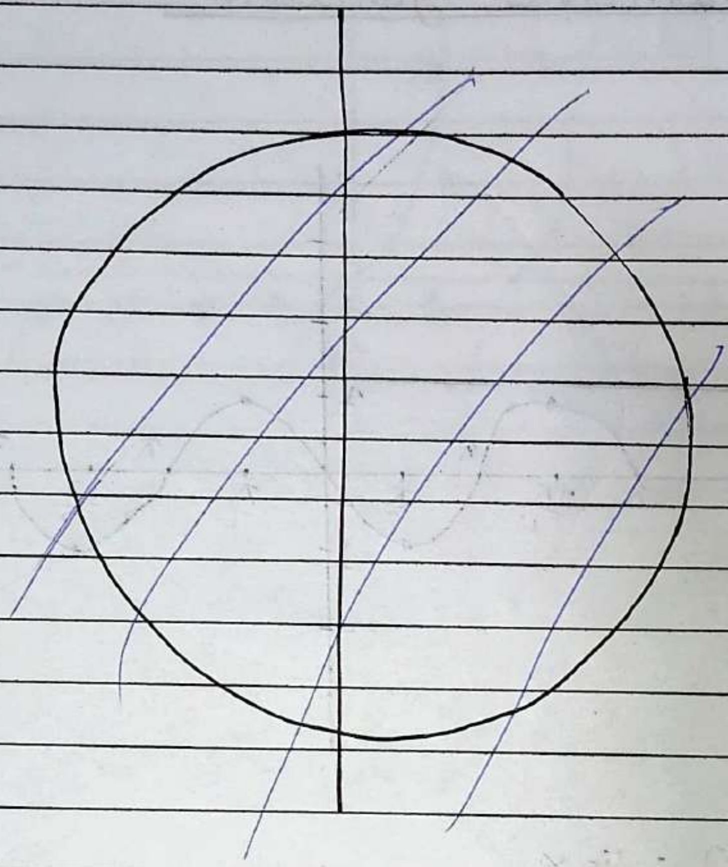
Anticlockwise

$x \to$	0°	30° $\pi/6$	45° $\pi/4$	60° $\pi/3$	90° $\pi/2$	180° π	270° $3\pi/2$	360° 2π
Sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞
~~Cosec~~								

~~Sec~~ α	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
Sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
Cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞

Co-ordinate
Matrix
Determinant
Statistics
Straight line.
Trigonometry.

Sign of Trigonometric functions



2nd quadrant
(Sin and cosec are +ve)
(Rest all -ve)

1st quadrant
(All +ve)

+ve

-ve

+ve

(tan x and cot x are +ve)
(Rest all -ve)

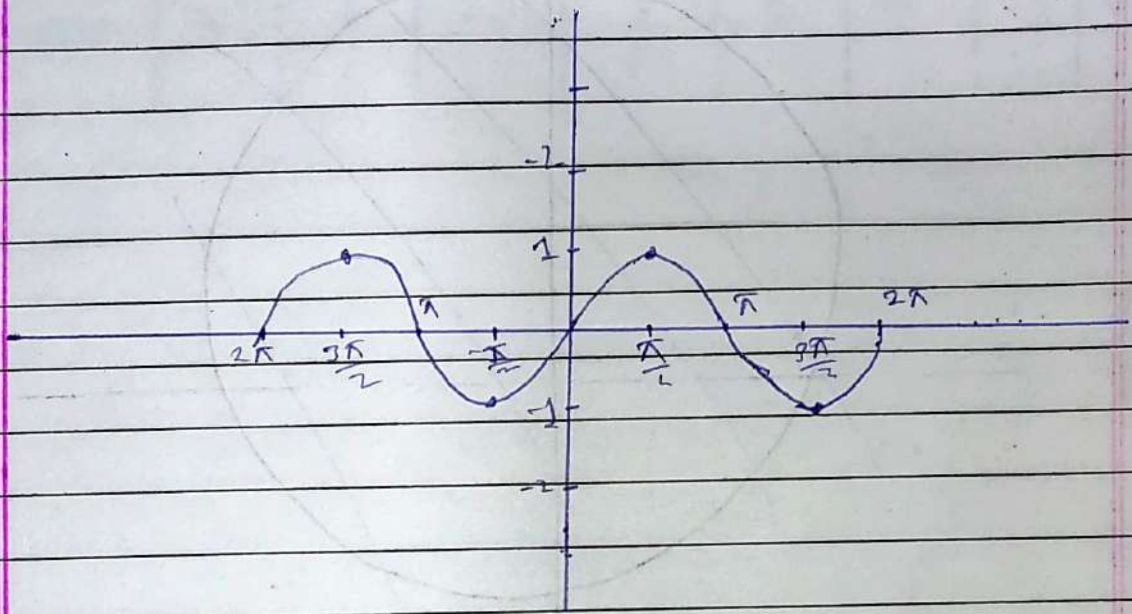
3rd quadrant

-ve

(cos x and sec x are +ve)
(Rest all -ve)

4th quadrant.

Range ~~and~~, domain and graph of trigonometric functions

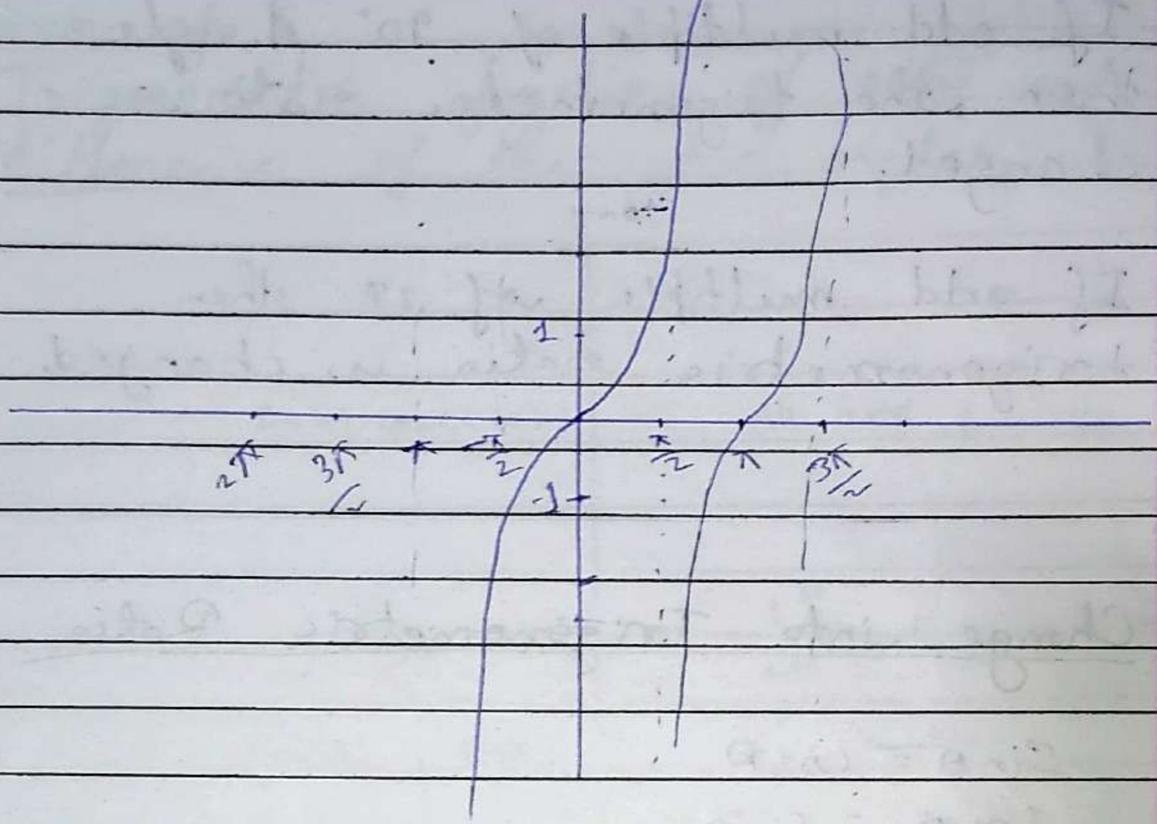


$F(x) = \sin x$

Domain $= R$

Range $= [-1, 1]$.

x



Domain ⇒ Real No. $- \left\{ (2n+1)\frac{\pi}{2} \right\}$

Range ⇒ $[-\infty, \infty]$ ~~$[-1 < x < 1]$~~

<u>**Note:**</u>

① If ~~any~~ ~~no~~ even multiple of 90° angle ~~then~~ the trigonometric ratio is not changed.

or.

If even multiple of 2π then the trigonometric ratio is not changed.

(7) If odd multiple of 90° ~~of~~. angle then the trigonometric ratio is changed.

or,

If odd multiple of 2π then trigonometric ratio is changed.

Change into Trigonometric Ratio

$\sin\theta = \cos\theta$

$\cos\theta = \sin\theta$

$\tan\theta = \cot\theta$

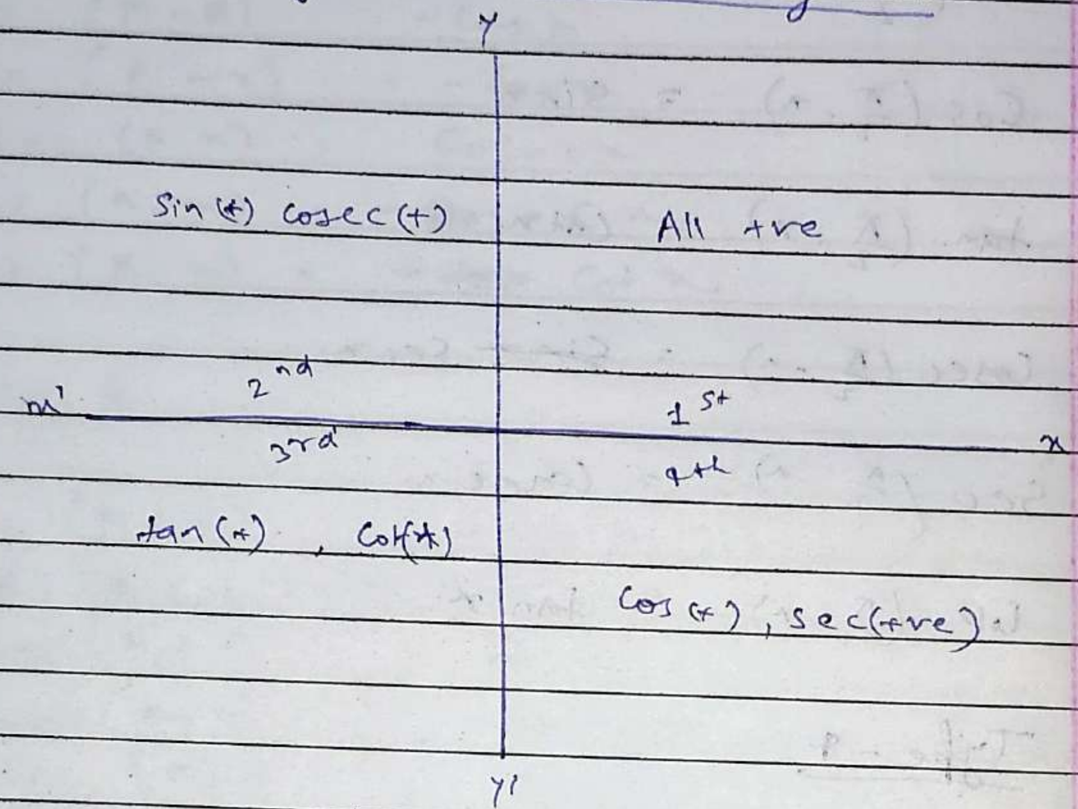
$\cot\theta = \tan\theta$

$\sec\theta = \operatorname{cosec}\theta$

$\operatorname{cosec}\theta = \sec\theta$.

~~Sum of~~

Trigonometric functions of ~~some~~ sum and difference of the two angles.



Type - 1

$\sin(-x) = \sin(0-x) = -\sin x.$
$\cos(-x) = \cos(0-x) = +\cos x$
$\tan(-x) = \tan(0-x) = -\tan x$
$\cot(-x) = \cot(0-x) = -\cot x$
$\sec(-x) = \sec(0-x) = +\sec x$
$\operatorname{cosec}(-x) = \operatorname{cosec}(0-x) = -\operatorname{cosec} x.$

Type 2

$\sin(0+x) = \sin x.$
$\cos(0+x) = \cos x$
$\tan(0+x) = \cancel{\cot x}\ \tan x.$
$\operatorname{cosec}(0+x) = \operatorname{cosec} x$
$\sec(0+x) = \sec x.$
$\cot(0+x) = \cot x.$

Type - 3

$\sin\left(\frac{\pi}{2} - x\right) = \cos x.$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x.$

$\mathrm{cosec}\left(\frac{\pi}{2} - x\right) = \cancel{\sin x}\ \sec x$

$\sec\left(\frac{\pi}{2} - x\right) = \mathrm{cosec}\, x$

$\cot\left(\frac{\pi}{2} - x\right) = \tan x.$

Type - 4

$\sin\left(\frac{\pi}{2} + x\right) = -\cos x.$

$\cos\left(\frac{\pi}{2} + x\right) = \sin x.$

$\tan\left(\frac{\pi}{2} + x\right. = -\cot x.$

$\mathrm{cosec}\left(\frac{\pi}{2} + x\right) = -\sec x$

$\sec\left(\frac{\pi}{2} + x\right) = \mathrm{cosec}\, x.$

$\cot\left(\frac{\pi}{2} + x\right) = -\tan x.$

Type - 5

$\sin(\pi - x) = \sin x.$

$\cos(\pi - x) = -\cos x$

$\tan(\pi - x) = -\tan x$

$\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$

$\sec(\pi - x) = -\cancel{\text{co}}\sec x$

$\cot(\pi - x) = \cancel{-\tan}\ \cot x.$

Type 6.

$\sin(\pi + x) = -\sin x.$

$\cancel{\cos}\ (\cancel{\pi}\ \cos(\pi + x) = -\cos x.$

$\tan(\pi + x) = \tan x.$

$\operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$

$\sec(\pi + x) = -\sec x$

$\cot(\pi + x) = \cot x.$

Type 7.

$\sin\left(\frac{3\pi}{2} \cancel{+} - x\right) = -\cos x$

$\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$

$\tan\left(\frac{3\pi}{2} - x\right) = +\cot x$

$\operatorname{cosec}\left(\frac{3\pi}{2} - x\right) = -\cancel{\tan x}\ \sec x$

$\sec\left(\frac{3\pi}{2} - x\right) = -\operatorname{cosec} x$

$\cot\left(\frac{3\pi}{2} - x\right) = \tan x.$

Type 8.

$\sin\left(\frac{3\pi}{2}+x\right) = \cos x.$

$\cos\left(\frac{3\pi}{2}+x\right) =$

Type 9.

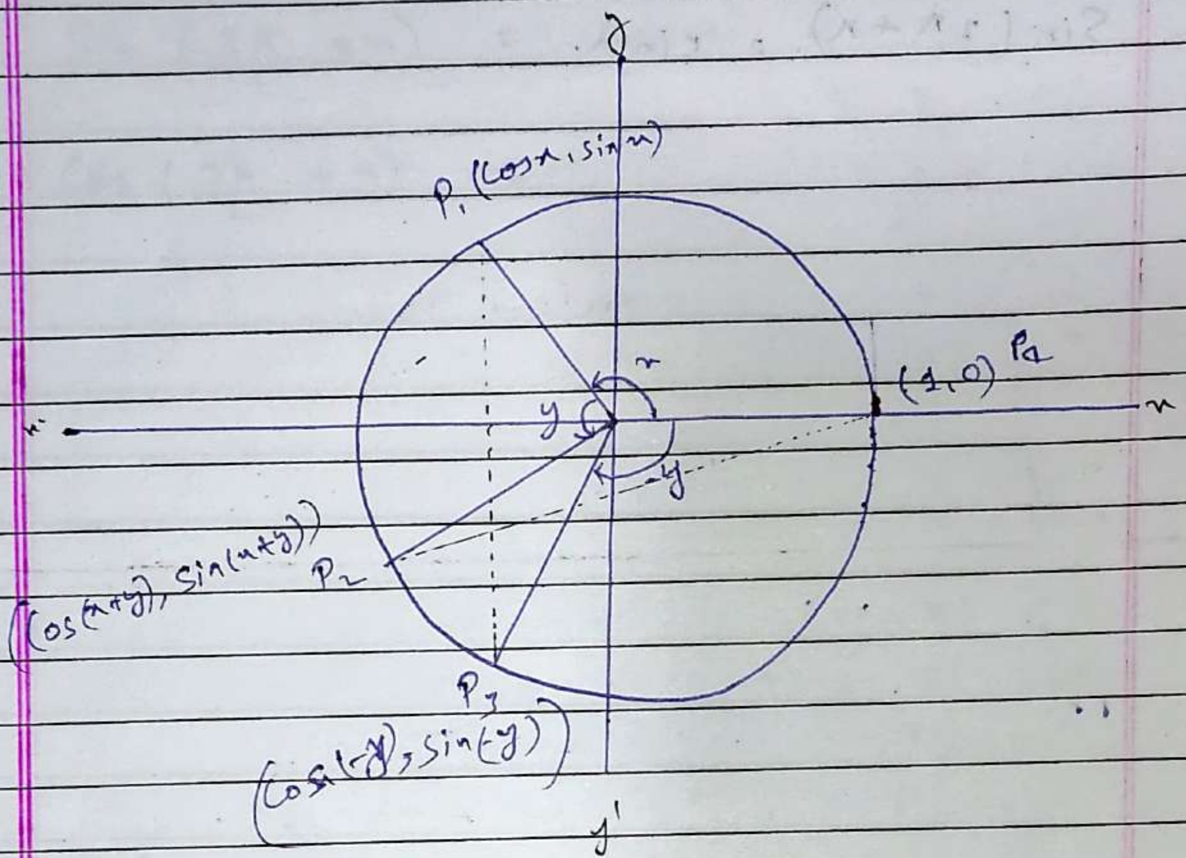
$\sin(2\pi - x) = -\sin x.$

$\cos(2\pi - x) = +\cos x.$

Type 10.

$\sin(2\pi + x) = \sin x.$

1. $\boxed{\cos(x+y) = \cos x . \cos y - \sin x . \sin y}$



Let point $P_1(\cos x, \sin y)$, $P_2[\cos(x+y), \sin(x+y)]$, $P_3[\cos(-y), \sin(-y)]$, and $P_4[1, 0]$.

Triangle $\Delta P_1 O P_3$ and $\Delta P_2 O P_4$ are congruent then $P_1 P_3$ and $P_2 P_4$ are equal.

According to distance formula

$$P_1 P_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$P_1P_3 = \sqrt{[\cos(-y) - \cos x)]^2 + [\sin(-y) - \sin x)^2}$$

$$P_1P_3 = \sqrt{(\cos y - \cos x)^2 + [-\sin y - \sin x]^2}$$

Sq^n on both sides, we have.

$$(P_1P_3)^2 = (\cos y - \cos x)^2 + (\sin y + \sin x)^2$$

$$= \cos^2 y + \cos^2 x - 2\cos x.\cos y + \sin^2 y + \sin^2 x + + 2\sin x.\sin y.$$

$$(P_1P_3)^2 = 1 + 1 + 2(\sin x \sin y - \cos x.\cos y)$$

$$= 2 + 2(\sin x.\sin y - \cos x.\cos y) \quad \text{— (i)}$$

Again,

$$P_2P_4 = \sqrt{[1 - \cos(x+y)]^2 + [0 - \sin(x+y)]^2}$$

Sq^2 on both sides.

$$(P_2P_4)^2 = 1 + \cos^2(x+y) - 2\cos(x+y) + \sin^2(x+y).$$

$$= 2 - 2\cos(x+y) \quad \text{— (ii)}$$

From ① and ②, we have.

$$(P_1 P_3)^2 = (P_2 P_4)^2$$

$\Rightarrow 2 + 2(\sin x \sin y - \cos x \cos y) = 2 - 2\cos(x+y)$

$$\boxed{\cos(x+y) = \cos x \cos y - \sin x \cdot \sin y}$$

Proved.

② $\boxed{\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y}$

Limits And Derivatives

New CH

Limit :

We say limit $x \to a^-$. $F(x)$ is the expected value of F at $x = a$ given the value of F near x to the left of a. This value is called the left hand limit of $F(x)$ at a

We say limit $x \to a^+$ $F(x)$ is the expected value of F at $x = a$ given the value of F near x to the right of a. This value is called the right hand limit of $F(x)$ at a.

If the right hand, left hand limit are equal to each other then the limit exists.

We called at common value as the limit of $F(x)$ at $x = a$, and it is denoted by $\lim_{x \to a} F(x)$.

Algebra of limits.

Let f and g be two functions such that both $\lim_{x \to a} F(x)$ and $\lim_{x \to a} g(x)$

exists then

i) $\lim_{x \to a} [F(x) + g(x)] = \lim_{x \to a} F(x) + \lim_{x \to a} g(x)$

ii) $\lim_{x \to a} [F(x) - g(x)] = \lim_{x \to a} F(x) - \lim_{x \to a} g(x)$

iii) $\lim_{x \to a} [F(x) \cdot g(x)] = \lim_{x \to a} F(x) \cdot \lim_{x \to a} g(x)$

iv) $\lim_{x \to a} \left[\frac{F(x)}{g(x)}\right] = \frac{\lim_{x \to a} F(x)}{\lim_{x \to a} g(x)}$

Q. If $f(x) = 2x + 1$ where x belongs to $x \in R$ are find $\lim_{x \to 1} f(x)$.

then check the lim.

Soln

$$LHL = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (2x+1)$$

$$= 2(1) + 1$$

$$= 3.$$

$$RHL = \lim_{x \to 1^+} F(x)$$

$$= \lim_{x \to 1} (2x+1)$$

$$= 2+1$$

$$= 3.$$

$\because$ $LHL = RHL$

$\therefore$ Then the limit exists.

VGC

★ For any positive integer n

then $\boxed{\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}}$

★ Limit of trigonometric functions

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

(ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$

Example:

VVI

Q. $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$

$\Rightarrow \lim_{x \to 1} \frac{x^{15} - 1^{15}}{x - 1} \cdot \frac{x - 1}{x^{10} - 1^{10}}$

$\Rightarrow \lim_{x \to 1} \frac{x^{15} - 1}{x - 1} \div \frac{x^{10} - 1}{x - 1}$

$\Rightarrow \lim_{x \to 1} \frac{x^{15} - 1}{x - 1} \div \lim_{x \to 1} \frac{x^{10} - 1}{x - 1}$

$= 15 . (1)^{15-1} \div (10) . (1)^{10-1}$

$= 15 \div 10 = \frac{15}{10} = \frac{3}{2}$ Ans

Q. Evaluate $\lim_{x \to 0} \frac{\sin 4x}{\sin 2x}$.

$\Rightarrow \lim_{x \to 0} \frac{\sin 4x}{\sin 2x}$

$\rightarrow \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{2x}{\sin 2x} \cdot 2$

$\rightarrow 2\left[\lim_{x \to 0} \frac{\sin 4x}{4x} \div \frac{\sin 2x}{2x}\right]$

$\rightarrow 2\left[\lim_{x \to 0} \frac{\sin 4x}{4x} \div \lim \frac{\sin 2x}{2x}\right]$

$\rightarrow 2.(1 \div 1.) \Rightarrow 2.$ Ans.

Q. Evaluate $\lim_{x \to 0} \frac{\tan x}{x}$

$\Rightarrow \lim_{x \to 0} \frac{\tan x}{x} \Rightarrow \lim_{x \to 0} \frac{\sin x}{\cos x} \cdot x.$

Ex: 13.1

1. $\lim_{x \to 3} x+3 \Rightarrow 3+3 = 6$ Ans

(4) $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4} \Rightarrow \frac{3.4 - 2 - 10}{4 - 4} = \frac{0}{0} = \infty.$

Now, $\lim_{x \to 2} \frac{3x^2 - 6x + 5x - 10}{(x+2)(x-2)}$

$\Rightarrow \lim_{x \to 2} \frac{3x(x-2) + 5(x-2)}{(x+2)(x-2)}$

$\Rightarrow \lim_{x \to 2} \frac{(x-2)(3x+5)}{(x+2)(x-2)} = \frac{11}{4}$ Ans

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Ex 13.1

(8) $\lim_{x \to 3} \frac{x^4 - 81}{2x^3 - 5x - 3}$

$\Rightarrow \lim_{x \to 3} \frac{(x^2)^2 - 9^2}{2x^2 - 6x + x - 3}$

$\Rightarrow \lim_{x \to 3} \frac{(x^2 + 9)(x^2 - 9)}{2x(x - 3) + 1(x - 3)}$

$\Rightarrow \lim_{x \to 3} \frac{(x^2 + 9)(x^2 - 3^2)}{(x - 3)(2x + 1)}$

$\Rightarrow \lim_{x \to 3} \frac{(x^2 + 9)(x + 3)(x - 3)}{(x - 3)(2x + 1)}$

$= \frac{108}{7}$ Ans

(15) $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi - x}$

$\Rightarrow \frac{1}{\pi} \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi - x}$

$= \frac{1}{\pi} \cdot 1$

$= \frac{1}{\pi}$ Ans

(13) $\lim_{x \to 0} \frac{\sin ax}{bx} \times \frac{ax}{ax} = \lim_{x \to 0} \frac{\sin ax}{ax} \cdot \frac{a}{b}$

$= \frac{a}{b} \lim_{x \to 0} \frac{\sin ax}{ax}$

$= \frac{a}{b} \cdot 1 = \frac{a}{b}$ Ans.